PROBLEM 1. Consider transmission over an ISI channel where the output after matched filtering is

\[ U[n] = \sum_{k=-\infty}^{\infty} d[k] I[n-k] + V[n] \]

and \( d[k] \) is given by

\[ d[k] = \begin{cases} 2^{-k|k|} & \text{if } k \text{ is odd}, \\ \frac{5}{2} \cdot 2^{-k|k|} & \text{if } k \text{ is even}, \end{cases} \]

and, as we already know from the lecture, the noise \( V[n] \) has autocorrelation function \( R_V[k] = \frac{N_0}{2} d[-k] \). Find the filter \( d_W[k] \) to whiten the noise. Choose the whitening filter such that the resulting communication channel after the whitening filter is causal.

PROBLEM 2. Consider the discrete-time equivalent of a band-limited AWGN channel with noise power \( N_0/2 = 1 \). The equivalent channel is a 3-tap channel defined as

\[ U[n] = \beta I[n+1] + \alpha I[n] + \beta I[n-1] + V[n]. \]

The Gaussian noise \( V[n] \) has autocorrelation function

\[ R_V[k] = d[-k] = \alpha \delta[k] + \beta \delta[k-1] + \beta \delta[k+1], \]

and \( \alpha \) and \( \beta \) are positive real coefficients satisfying \( \alpha^2 > 4 \beta^2 \). We also assume that the information symbols \( I[n] \) are i.i.d. random variables with mean 0 and variance \( \mathcal{E} \).

(a) Use the zero-forcing equalizer to remove all of the inter-symbol interference. Find the frequency response of the filter and calculate the variance of the effective noise \( \tilde{V}[n] = I[n] - \hat{I}_{ZF}[n] \) (see Equation (4.13) of your lecture notes).

(b) Now apply the LMMSE approach. Find the frequency response of LMMSE filter \( A_{LMMSE}(f) \). Calculate the resulting noise variance \( \sigma_{LMMSE}^2 \) (as in Equation (4.15) of your lecture notes) using the following steps:

(i) Show that, in general, \( \sigma_{LMMSE}^2 = \mathbb{E}[I[n]^2] - \mathbb{E}[\hat{I}_{LMMSE}[n]I[n]] \).

(ii) Use (i) to find \( \sigma_{LMMSE}^2 \) for the particular example considered in the problem.

(c) Another tempting receiver the matched filter: It passes the channel output signal through a filter whose impulse response is the time-reversed version of the original channel impulse response. Show that the corresponding output signal can be expressed as

\[ I_{MF}[n] = g[0] I[n] + \sum_{k \neq 0} g[k] I[n-k] + \sum_{m} f[m] V[n-m], \]

give the values of the coefficients \( g[k] \), and \( f[m] \), and calculate the effective noise variance, that is, the variance of the \( \frac{1}{g[0]} I_{MF}[n] - I[n] \).
(d) For \( \alpha = 1 \) and \( \beta = 0.4 \), plot the effective noise variances you found in (a)–(c) for the range of \( E \in [-20 \, \text{dB}, +20 \, \text{dB}] \) and compare them.

*Hint.* You may need the following integrals: For \( a \geq |b| \),
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{a^2 - b^2}}, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dx}{(a + b \cos x)^2} = \frac{a}{(a^2 - b^2)^{3/2}}.
\]

**Problem 3.** Consider the following two user discrete-time channel
where \( I[n] \) is the information symbol from user 1 and \( J[n] \) is the information symbol from user 2. We assume that the sequences \( I[n] \) and \( J[n] \) are both i.i.d. zero-mean random variables with unit variance independent of each other, and that \( W[n] \) is white Gaussian noise with variance \( N_0/2 = 1 \). Given the channel output \( Y[n] \) we would like to design linear MMSE estimators to detect both \( I[n] \) and \( J[n] \) in the form
\[ \hat{I}[n] = a_{-1}Y[n+1] + a_0Y[n] + a_1Y[n-1] \quad \text{and} \quad \hat{J}[n] = b_{-1}Y[n+1] + b_0Y[n] + b_1Y[n-1]. \]
Compute the optimal coefficients \( a_i, b_i \) for \( i = -1, 0, 1 \) which minimize the mean squared error \( \mathbb{E}[(\hat{I}[n] - I[n])^2] \) and \( \mathbb{E}[(\hat{J}[n] - J[n])^2] \).

**Problem 4.** Consider the noisy ISI channel given by
\[ Y[n] = X[n] + X[n-1] + Z[n] \]
where \( X[n] \) and \( Y[n] \) are a sequence of i.i.d. Gaussian random variables, with zero mean and unit variance and \( X[n] \in \{-1, 1\} \) uniformly. Calculate the symbol-wise MAP estimate of \( X[1], \ldots, X[4] \) using the BCJR algorithm, if the received sequence is \( 1, -2, -1, 3, 2 \). Assume that the channel is in state +1 at the beginning and at the end of the sequence (i.e., \( X[0] = X[5] = +1 \)). Compare this to the decoding estimate from MLSE (Viterbi) decoder.

**Problem 5.** Consider the following real channel,
\[ Y = hX + Z, \]
where \( X \in \mathbb{R} \) is a random variable, with \( \mathbb{E}[X] = 0 \) and \( \mathbb{E}[X^2] = E \), \( h \) is a fixed real (column) vector, and \( Z \) is a zero-mean random vector with covariance matrix \( I \), chosen independently of \( X \).

(a) An estimator \( \hat{X}(y) \) is said to be *unbiased* if \( \mathbb{E}[\hat{X}(Y)|X = x] = x \).

(i) What is the constraint for a linear estimator, i.e \( \hat{X} = a^T Y \) to be unbiased?

(ii) Find the unbiased linear estimator that minimizes the mean squared error \( \sigma^2_{\text{unbiased}} = \mathbb{E}[(X - \hat{X})^2] \) and the value of \( \sigma^2_{\text{unbiased}} \) for this estimator.

*Hint.* By Cauchy–Schwartz inequality, \( (a^T a)(h^T h) \geq |a^T h|^2 \).

(b) In this part, we don’t restrict ourselves to unbiased estimators. Suppose \( \hat{X} = a^T Y \) is a linear estimator. Find the linear estimator that minimizes the mean squared error \( \sigma^2 = \mathbb{E}[(X - \hat{X})^2] \) and the value of \( \sigma^2 \) for this estimator.

*Hint.* First assume \( a^T h = c \), and minimize \( \sigma^2 \) with respect to the vector \( a \) with the constraint \( a^T h = c \), and then minimize the result with respect to \( c \).

(c) Compare the two “signal to noise ratio”s \( E/\sigma^2_{\text{unbiased}} \) and \( E/\sigma^2 \).

(d) Now assume \( X \) is equally likely to be +1 or -1. Suppose a decision is made by quantizing the estimate \( \hat{X} \) from either part (a) or (b) to \( \pm 1 \). Which estimator would you choose to minimize the probability of error?