ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 28 — Advanced Digital Communications

Homework 12 — Dec. 12, 2016

PROBLEM 1.

(a) Plot the fixed points of density evolution (as a function of ϵ) and determine the BP thresholds $\epsilon^*_{\mathrm{BP}}(l, r)$ for the regular $(2, 4)$, $(3, 6)$, $(4, 8)$, and $(5, 10)$ ensembles.

(b) You observed in part (a) that even though in all cases the code rate was $\frac{1}{2}$ and, in principle, one has to be able combat an erasure rate of up to 0.5, the BP thresholds were way below 0.5. As we have seen in the class, we can use irregular LDPC codes to do so. Consider the ensemble defined with the pair of degree distributions *from edges perspective*

$$\lambda(x) = \frac{2}{5}x + \frac{1}{5}x^2 + \frac{3}{25}x^3 + \frac{1}{5}x^6 + \frac{2}{25}x^7 \quad \text{and} \qquad \rho(x) = x^5.$$

(i) What is the rate of this ensemble?

(ii) Plot the fixed points of density evolution and determine the BP threshold for this irregular ensemble.

PROBLEM 2. Recall that a binary erasure channel (BEC) W_1 with erasure probability ϵ_1 is better than the BEC W_2 with erasure probability ϵ_2 if and only if $\epsilon_1 \le \epsilon_2$. In this case we write $W_1 \succeq W_2$ to denote this ordering. Recall also that both 'minus' and 'plus' channels synthesized by the application of polar transform to a BEC are themselves BECs.

(a) Prove that if $W_1 \succeq W_2$ then $W_1^- \succeq W_2^-$ and $W_1^+ \succeq W_2^+$.

(b) Using (a), show that if $W_1 \succeq W_2$ then, for any sign sequence $(s_1, \dots, s_n) \in \{-, +\}^n$, $W_1^{s_1 \cdots s_n} \succeq W_2^{s_1 \cdots s_n}$.

(c) We know from the class that polar codes are channel specific: A polar code designed for channel W_1 may not be suitable for communication over the channel W_2. Suppose we design a polar code for communication over a binary erasure channel with erasure probability ϵ. But, in practice the channel turns out to behave better and have erasure probability $\epsilon' < \epsilon$. What can you say about the performance of this mismatched code?

PROBLEM 3. Let $s_1, \dots, s_n$ be a sequence of length n of 'plus' and 'minuses' that contains m '$-$'s and $n - m$ '$+$'s. Note that in general there are $\binom{n}{m}$ such sequences each of them defines one synthetic channel $W^{s_1 \cdots s_n}$ obtained by n-fold application of polar transform to a BEC with erasure probability ϵ. Prove that among these channels, the one indexed by

$$(s_1, s_2, \dots, s_n) = (\underbrace{-, -, \dots, -}_{m}\underbrace{+, +, \dots, +}_{n-m})$$

is the worst while the one indexed by

$$(s_1, s_2, \dots, s_n) = (\underbrace{+, +, \dots, +}_{n-m}, \underbrace{-, -, \dots, -}_{m})$$

is the best channel.

Hint. Start by looking at two channels indexed by sign sequences containing the same number of plus and minuses and differing only in two consecutive positions.

PROBLEM 4. In the class we mentioned that the block error probability of polar codes decay roughly like $2^{-\sqrt{N}}$ (more precisely 2^{-N^β} for any $\beta < \frac{1}{2}$). In this problem we will show that this bound is exponentially tight. Namely, we cannot get a decay rate of faster than $2^{-\sqrt{N}}$ in the block length. As usual, we focus on the case of erasure channel.

(a) Convince yourself that if $P_\mathrm{e}(\mathcal{A}_n)$ denotes the block-error probability of a polar code of block length $N = 2^n$ defined with information indices $\mathcal{A}_n \in \{-,+\}^n$,

$$P_\mathrm{e}(\mathcal{A}_n) \ge \max_{(s_1,\dots,s_n)\in\mathcal{A}_n} \epsilon(s_1,\dots,s_n)$$

where $\epsilon(s_1,\dots,s_n)$ is the erasure probability of the synthetic channel $W^{s_1\cdots s_n}$.

(b) Fix a sign sequence $(s_1,\dots,s_n)$ and define two sequences a_ℓ and b_ℓ as follows:

$$a_0 = b_0 = \epsilon,$$

$$a_\ell = \begin{cases} 2a_{\ell-1} - a_\ell^2 & \text{if } s_\ell = - \\ a_\ell^2 & \text{if } s_\ell = + \end{cases} \qquad b_\ell = \begin{cases} b_{\ell-1} & \text{if } s_\ell = - \\ b_\ell^2 & \text{if } s_\ell = +. \end{cases}$$

Note that $\epsilon(s_1,\dots,s_n) = a_n$.

(i) Prove that $\forall \ell = 0, 1, \dots, n$, $a_\ell \ge b_\ell$.

(ii) Show that i f p is the number of 'pluses' in $s_1,\dots,s_n$, then

$$b_n = \epsilon^{(2^p)}$$

(iii) Conclude that $a_n \le 2^{-N^\beta}$, $N = 2^n$, implies

$$p \ge \beta n - c(\epsilon)$$

where $c(\epsilon)$ is a constant that depends only on ϵ.

(c) Note that the number of sign sequences of length n with at least p 'pluses' equals

$$\sum_{j=p}^{n} \binom{n}{j}.$$

(i) Show that for any $z \in [0:1]$, $\binom{n}{j} \le z^{-j}(1-z)^{-(n-j)}$.

(ii) From (i) conclude that

$$\binom{n}{j} \le 2^{nh_2(j/n)}; \qquad h_2(x) := x\log_2\frac{1}{x} + (1-x)\log_2\frac{1}{1-x}$$

It is easy to see that $h_2(x) \le 1$ with equality iff $x = \frac{1}{2}$ and is increasing for $x \in [0:1/2]$ and decreasing for $x \in [1/2:1]$ (e.g., by plotting the function).

(iii) Using the fact that the binomial coefficient $\binom{n}{j}$ is decreasing in j for $j \ge n/2$, prove that, if $p \ge n/2$,

$$\sum_{j=p}^{n} \binom{n}{j} \le n2^{nh_2(p/n)}$$

(iv) Conclude that if $\beta > \frac{1}{2}$, the *fraction* of sign sequences with more than $\beta n - c(\epsilon)$ 'pluses' vanishes as $n \to \infty$.

(d) Using (b) and (c) conclude that the fraction of synthetic channels whose erasure probabilities decay like 2^{-N^β}, $\beta > \frac{1}{2}$, goes to zero as $n \to \infty$.

(e) Finally using (a) conclude that unless the code rate vanishes, the block error probability of a sequence of polar codes cannot decay faster than $2^{-\sqrt{N}}$.