ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 24 Advanced Digital Communications
Homework 10 (*Graded, Due Dec. 5, 2016*) Nov. 28, 2016

PROBLEM 1. (25 pts) Consider a scalar, real-valued two-user situation:

$$Y = 10x_A + x_B + Z,$$

where x_A and x_B are independently and uniformly selected from $\pm\sqrt{\mathcal{E}}$, as usual, and the noise Z is the usual AWGN of variance $N_0/2$.

(a) Describe the ML decoding regions and compute its error probability $P_{\mathrm{e}}^{\mathrm{ML}}$.

Next, we want to study a suboptimal detector:

(b) We first decode x_A simply by thresholding the signal Y at zero. That is,

$$\hat{x}_A = \begin{cases} \sqrt{\mathcal{E}}, & \text{if } Y \geq 0, \\ -\sqrt{\mathcal{E}}, & \text{if } Y < 0. \end{cases}$$

Compute the average error probability $P_{\mathrm{e}}^{(A)}$ of this decision (in terms of the Q-function).

(c) Continuing with the detector from (b), we next form

$$\tilde{Y} = Y - 10\hat{x}_A,$$

where $\hat{x}_A$ is precisely the decision we took in (b) (which may be right or wrong). Engineers refer to this as *canceling the interference* created by user A. Then, we decide for user B:

$$\hat{x}_B = \begin{cases} \sqrt{\mathcal{E}}, & \text{if } \tilde{Y} \geq 0, \\ -\sqrt{\mathcal{E}}, & \text{if } \tilde{Y} < 0. \end{cases}$$

Unfortunately computing the probability of error for the second user, that is $\Pr\{\hat{x}_B \neq x_B\}$ is difficult. Instead, consider a *genie-aided* decoder for user (B) that, instead of $\tilde{Y}$, uses

$$Y' = Y - 10x_A$$

as the input to the decision box and decides

$$\tilde{x}_B = \begin{cases} \sqrt{\mathcal{E}}, & \text{if } Y' \geq 0, \\ -\sqrt{\mathcal{E}}, & \text{if } Y' \leq 0 \end{cases}$$

Compute the average error probability $P_{\mathrm{e}}^{(B)}$ for this *genie-aided* decoder (in terms of the Q-function).

(d) The overall detector resulting from (b) and (c) is referred to as a *successive decoder.* Show that an overall decoding error event happens if and only if the *genie-aided* decoder makes an error. Namely

$$\{(\hat{x}_A, \hat{x}_B) \neq (x_A, x_B)\} = \{(\hat{x}_A, \tilde{x}_B) \neq (x_A, x_B)\}.$$

(e) Give simple upper *and lower* bounds on the overall probability of the successive decoder in terms of $P_e^{(A)}$ and $P_e^{(B)}$.

(f) Let us call a detector *high-SNR optimal* if it satisfies

$$\lim_{\mathcal{E}\to\infty} \frac{P_e}{P_e^{\mathrm{ML}}} = 1,$$

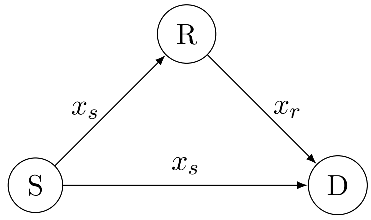
where P_e is the error probability of the detector in question, and P_e^{ML} is the error probability of the ML detector (as in (a)). Prove that the successive decoder you analyzed in (b)–(d) is high-SNR optimal.

Hint. You may use the fact that for all $a > 1$, the ratio $\lim_{x\to\infty} Q(ax)/Q(x) = 0$.

PROBLEM 2. (15 pts) Consider a network with three nodes: a source node, a destination node, and a relay node described via

$$\begin{aligned} Y_r[n] &= H_{sr}[n]x_s[n] + Z_r[n], \\ Y_d[n] &= H_{sd}[n]x_s[n] + H_{rd}[n]x_r[n] + Z_d[n], \end{aligned}$$

where $Z_r[n]$ and $Z_d[n]$ are i.i.d. circularly-symmetric complex Gaussian noise processes of variance N_0, $H_{sr}[n]$, $H_{sd}[n]$ and $H_{rd}[n]$ are independent Rayleigh fading coefficients for the source–relay, source–destination, and relay–destination links, respectively, known to the receiver (but not to the transmitter) and $x_s[n]$ and $x_r[n]$ are the sent signals from the source and the relay respectively. Clearly, the diversity order on the link from the source to the destination is simply 1. We wonder if the relay can increase the diversity orders?



Consider the *amplify-and-forward* transmission scheme: At time $n = 1$, the relay remains silent, hence the receiver observes

$$Y[1] = H_{sd}[1]x_s[1] + Z_d[1].$$

In the next time-slot, $n = 2$, the source node is silent and the relay node transmits what it has received, $x_r[2] = Y_r[1]$, thus the destination node receives

$$Y_d[2] = H_{rd}[2]x_r[2] + Z_d[2] = H_{rd}[2]H_{sr}[1]x_s[1] + H_{rd}[2]Z_r[1] + Z_d[2].$$

Find the resulting overall diversity order when the destination uses optimal detection based on both received samples $Y_d[1]$ and $Y_d[2]$ for the following two cases:

(a) $H_{rd}[2] = h$, where h is a complex constant and $|h| > 0$.

(b) $H_{rd}[2]$ is Rayleigh fading independent of $H_{sr}[1]$ and $H_{sd}[1]$.

Note. You may assume that BPSK is used if this helps. As you may have noticed, the diversity order is independent of the modulation format — be it BPSK, 16-QAM, or your good old favorite.

PROBLEM 3. (15 pts) Consider a transmitter sending signals to be received by two receive antennas. However, due to an unfortunate coincidence, there is a flag fluttering in the wind quite close to one of the receive antennas and sometimes completely blocks the received signal.

(a) In the absence of the flag, the received signal is given by a flat fading model,

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} X + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

where Y_1, Y_2 are the received signals on first and second receive antennas respectively, X is the transmitted signal and H_1, H_2 are respectively the fading attenuation from the transmitter to the first and second receive antennas. Assume that X is binary, i.e., $X \in \{-\sqrt{\mathcal{E}}, \sqrt{\mathcal{E}}\}$. The additive noise Z_1, Z_2 are assumed to be independent circularly symmetric complex Gaussian with variance (each) of σ^2. Assume that H_1, H_2 are i.i.d. circularly-symmetric complex Gaussian with unit variance known to the receiver.

Compute an upper bound to the average the error probability and comment about the behavior of the error probability with respect to SNR for high SNR.

(b) Now consider the presence of fluttering flag which could potentially block only the second receive antenna. The model given in (a) now changes to:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ FH_2 \end{bmatrix} X + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

where the random variable F equals 0 if the flag obstructs the signal and 1 otherwise. Suppose due to the random fluttering, the flag blocks a fraction q of the transmissions, i.e., for a fraction q of the transmission, one receives only the signal from the first antenna. Assume the receiver knows F.

Conditioned on F, write down the error probabilities, i.e., find an expression for the error probability.

(c) Find the overall error probability in the presence of the fluttering. How does the error probability behave at high SNR?

PROBLEM 4. (20 pts) Consider the following communication setting, where there is not only additive noise but the phase of the transmitted complex vector $\boldsymbol{X}$ is corrupted as well:

$$\boldsymbol{Y} = e^{j\Theta}\boldsymbol{X} + \boldsymbol{Z},$$

where $\boldsymbol{Z}$ is a complex circularly symmetric Gaussian random vector with covariance matrix I and Θ is a uniform random variable in $[0, 2\pi]$. The random vector $\boldsymbol{Z}$ and the random variable Θ are independent of each other and also of $\boldsymbol{X}$. Suppose $\boldsymbol{X}$ is equally likely to be one of $\boldsymbol{x}_0$ or $\boldsymbol{x}_1$, where $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are vectors with equal energy, $\|\boldsymbol{x}_0\|^2 = \|\boldsymbol{x}_1\|^2 = \mathcal{E}$.

(a) Show that the conditional probability density $f(\boldsymbol{y}|\boldsymbol{X} = \boldsymbol{x}_m)$ is given by

$$f(\boldsymbol{y}|\boldsymbol{X} = \boldsymbol{x}_m) = Ce^{-\|\boldsymbol{y}\|^2} g(|\langle \boldsymbol{y}, \boldsymbol{x}_m \rangle|)$$

where $g(a) = (2\pi)^{-1} \int_0^{2\pi} \exp(2a\cos(\theta))\, d\theta$ and C is a constant.

Can the decoder make its decision on the basis of the two values $|\langle \boldsymbol{y}, \boldsymbol{x}_0 \rangle|$ and $|\langle \boldsymbol{y}, \boldsymbol{x}_1 \rangle|$ alone?

(b) Show that $g(a)$ is an increasing function for $a \geq 0$. Does the decision rule:

$$\text{“Choose } m \text{ that maximizes } |\langle \boldsymbol{y}, \boldsymbol{x}_m \rangle|\text{”}$$

minimize the probability of error?

(c) Assume further that $\langle \boldsymbol{x}_0, \boldsymbol{x}_1 \rangle = 0$. Find the error probability of the optimal decoder.

(d) Suppose now that the channel also corrupts the amplitude:

$$\boldsymbol{Y} = He^{j\Theta}\boldsymbol{X} + \boldsymbol{Z}$$

where H is an unknown scalar chosen independently of $\boldsymbol{X}$, $\boldsymbol{Z}$ and Θ. Would you spend any effort in estimating H? (A few of lines of explanation is sufficient.)

Problem 5. (25 pts) Consider transmission over two independent (parallel) channels

$$Y_1 = H_1X_1 + Z_1 \qquad \text{and} \qquad Y_2 = H_2X_2 + Z_2$$

where Z_1 and Z_2 are independent circularly symmetric Gaussian noises of mean zero and variance N_0 and H_1 and H_2 are independent Rayleigh fading coefficients, i.e., they are circularly symmetric complex Gaussian random variables with variance 1. We want to transmit *two* bits across a single use of two channels and we would like to compare the following two transmission strategies:

Strategy A Send independent BPSK symbols with power $\mathcal{E}$ across independent channels, that is we pick $X_1 \in \{\pm\sqrt{\mathcal{E}}\}$ and $X_2 \in \{\pm\sqrt{\mathcal{E}}\}$ independently.

Strategy B Map the 2 bits to a symbol from a symmetric 4-PAM constellation with power $\mathcal{E}$ and send *the same signal through both channels.* That is we set $X_1 = X_2 \in \{-3a, -a, a, 3a\}$.

(a) Compute the value of a so that the average power of the 4-PAM constellation is $\mathcal{E}$.

(b) Compute the error probability of each of the signaling strategies *conditioned on* $H_1 = h_1$ *and* $H_2 = h_2$ for arbitrary *complex-valued* constants h_1 and h_2.

(c) How does the (conditional) error probabilities in (b) behave when $\mathcal{E}$ gets large? Consider the cases $h_1 = h_2$ and $h_1 \neq h_2$ separately and decide which strategy is preferable in each case. (Note that the error probabilities you consider in this case describe the performance of the system without fading.)

(d) Compute the *average* error probabilities for each signaling strategy (averaged over the fading states H_1 and H_2 as usual).

(e) How do the average error probabilities behave when $\mathcal{E}$ is large? Which strategy is preferable?