Problem 1. Consider a signal $x[n]$, its Fourier transform $X(f)$, and its Z-transform $X(z)$.

(a) Assume $x[n]$ is conjugate-symmetric, that is, $x[n] = x^*[−n]$. Prove that $X(f)$ is real-valued. Furthermore, show that $X(z) = X^*(1/z^*)$.

(b) Consider signals $x[n]$ and $y[n]$ with Fourier transform $X(f)$ and $Y(f)$, respectively. The convolution of $x[n]$ and $y[n]$ is defined as

$$ (x * y)[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]. $$

Find the Fourier transform of $(x * y)[n]$ in terms of $X(f)$ and $Y(f)$.

(c) Define the signal $\bar{x}[n] = x^*[−n]$. Find the Fourier transform of $y$ defined below. Express the result in terms of $X(f)$. You may use the result from part (b).

$$ y[n] = (x * \bar{x})[n] = \sum_{k=-\infty}^{\infty} x[k]\bar{x}[n-k]. $$

(d) Show that the discrete Fourier transform can be written in terms of a matrix $F$ that we will refer to as the Fourier matrix. Give a general formula for the entries of the Fourier matrix and show that its inverse is simply $F^{-1} = F^H$. Any matrix that satisfies this relationship is called a unitary matrix. Finally, write out the Fourier matrix explicitly for dimensions 2, 3, and 4.

Problem 2. Let $X$ and $Y$ be discrete random variables defined on some probability space with a joint pmf $p_{XY}(x,y)$. Let $a, b \in \mathbb{R}$ be constants.

(a) Prove that $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$. Do not assume independence.

(b) Prove that if $X$ and $Y$ are independent random variables, then $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(c) Assume that $X$ and $Y$ are not independent. Find an example where $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$, and another example where $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(d) Prove that if $X$ and $Y$ are independent, then they are also uncorrelated, i.e.,

$$ \text{cov}(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0. $$

(e) Assume that $X$ and $Y$ are uncorrelated and let $\sigma_X^2$ and $\sigma_Y^2$ be the variances of $X$ and $Y$, respectively. Find the variance of $aX + bY$ and express it in terms of $\sigma_X^2, \sigma_Y^2, a, b$.

Hint. First show that $\text{cov}(X, Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$. 
Problem 3.
(a) For a non-negative integer-valued random variable \( N \), show that
\[
E[N] = \sum_{n>0} \Pr\{N \geq n\}.
\]
(b) Show, with whatever mathematical care you feel comfortable with, that for an arbitrary non-negative random variable \( X \),
\[
E[X] = \int_0^\infty \Pr\{X \geq a\} \, da.
\]
(c) Derive the Markov inequality, which says that for any non-negative random variable \( X \) and positive \( a \), we have
\[
\Pr\{X \geq a\} \leq \frac{E[X]}{a}.
\]
*Hint.* Sketch \( \Pr\{X \geq a\} \) as a function of \( a \) and compare the area of the rectangle with horizontal length \( a \) and vertical length \( \Pr\{X \geq a\} \) in your sketch with the area corresponding to \( E[X] \).
(d) Derive the Chebyshev inequality, which says that
\[
\Pr\{|Y - E[Y]| \geq b\} \leq \frac{\sigma_Y^2}{b^2}
\]
for any random variable \( Y \) with finite mean \( E[Y] \) and finite variance \( \sigma_Y^2 \).
(e) Derive the Chernoff bound, which says that for any random variable \( Z \),
\[
\Pr\{Z \geq b\} \leq E[e^{s(Z-b)}], \quad \forall s \geq 0.
\]
Problem 4. Let \( X_1, X_2, \ldots, X_n, \ldots \) be a sequence of independent identically distributed (i.i.d.) random variables with the common probability density function \( f_X(x) \). Note that \( \Pr\{X_n = \alpha\} = 0 \) for all \( \alpha \) and that \( \Pr\{X_n = X_m\} = 0 \), for \( m \neq n \).
(a) Find \( \Pr\{X_1 \leq X_2\} \). (Give a numerical answer, not an expression; no computation is required and a one- or two-line explanation should be adequate.)
(b) Find \( \Pr\{X_1 \leq X_2; X_1 \leq X_3\} \); in other words, find the probability that \( X_1 \) is the smallest of \( \{X_1, X_2, X_3\} \). (Again, think — do not compute.)
(c) Let the random variable \( N \) be the index of the first random variable in the sequence to be less than \( X_1 \); i.e.,
\[
\{N = n\} = \{X_1 \leq X_2; X_1 \leq X_3; \ldots; X_1 \leq X_{n-1}; X_1 > X_n\}.
\]
Find \( \Pr\{N \geq n\} \) as a function of \( n \).
(d) Show that \( E[N] = \infty \).
(e) Now assume that \( X_1, X_2, \ldots \) is a sequence of i.i.d. random variables each drawn from a finite set of values. Explain why you cannot find \( \Pr\{X_1 \leq X_2\} \) without knowing the pmf. Explain \( E[N] = \infty \).