ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 33 Advanced Digital Communications

Solutions to Final Exam Jan. 24, 2017

SOLUTION 1.

(a) The channel response is

$$D(z) = 2 + z + z^{-1}.$$

The LMMSE filter, by definition, minimizes the residual noise variance. Thus that would be our choice of equalizer. Its frequency response is obtained as follows. Since the information symbols are i.i.d., zero-mean, with unit variance and the noise spectrum is $S_V(z) = D(z)$,

$$D_{\text{LMMSE}}(z) = \frac{S_I(z)}{S_I(z)D(z) + 1} = \frac{1}{D(z) + 1} = \frac{1}{3 + z^{-1} + z}.$$

Replacing $z = e^{j2\pi f}$ we get

$$D_{\text{LMMSE}}(f) = \frac{1}{3 + 2\cos(2\pi f)}.$$

REMARK. Had we chosen the zero-forcing equalizer, the power spectrum of $\tilde{V}[n]$ would have been

$$S_{\tilde{V}}(f) = S_V(f)|D_{\text{ZF}}(f)|^2 = \frac{1}{2 + 2\cos(2\pi f)}.$$

We see that as f approaches $\pm\frac{1}{2}$, $S_{\tilde{V}}(f)$ blows up. Therefore, the noise variance is *infinity* at the output of the zero-forcing filter. (That's why zero-forcing is not always a good idea!)

(b) The spectrum of V can be factored as

$$S_V(z) = \underbrace{(1 + z^{-1})}_{A(z)}\underbrace{(1 + z)}_{A^*(1/z^*)}$$

The whitening filter that results in a causal effective channel is

$$D_W(z) = \frac{1}{A^*(1/z^*)} = \frac{1}{1 + z}$$

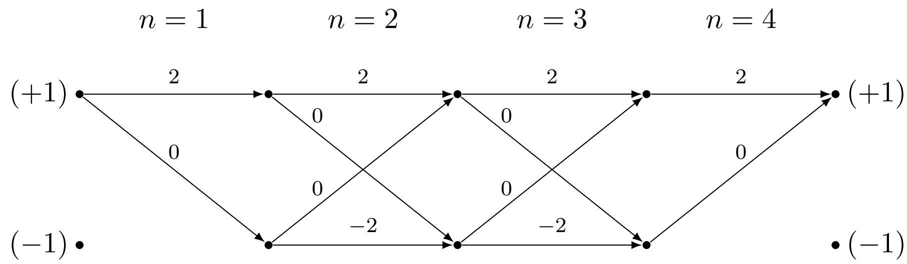
The output of the whitening filter is,

$$S[n] = d_W[n] * U[n] = \sum_{k \geq 0} I[n-k]f[k] + W[n]$$

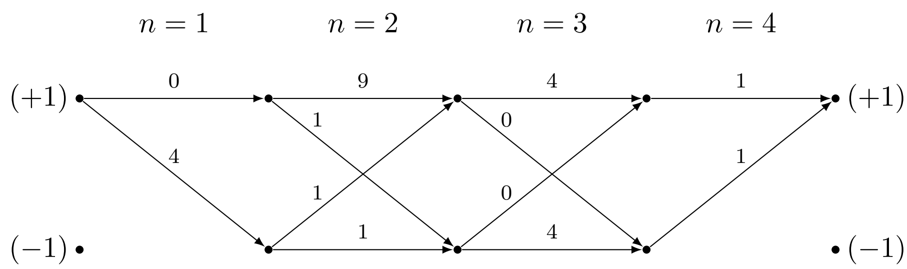
In the above, $W[n]$ is a circularly symmetric white Gaussian noise with variance 1 and $F(z) = D(z)D_W(z) = 1 + z^{-1}$. Therefore, the effective channel is

$$S[n] = I[n] + I[n-1] + W[n]$$

(c) (i) Here is the trellis with the edges labeled with the value of noiseless channel output, i.e. $I[n] + I[n-1]$,

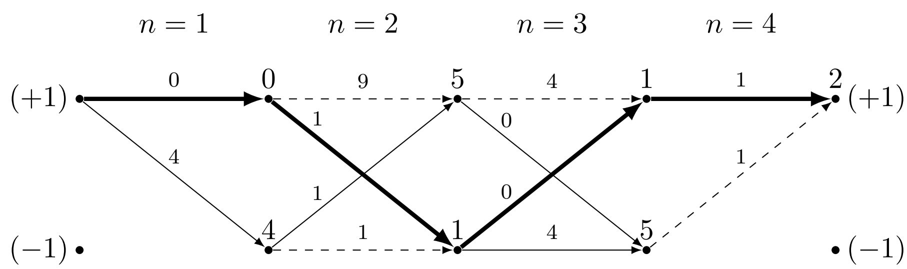


(ii) The Viterbi algorithm uses the path metric $|S[n] - (I[n] + I[n-1])|^2$ and finds the path with smallest metric (closest path to the output sequence) on the trellis. Given the output sequence $(2, -1, 0, 1)$ the edges will be labeled as follows:



$S[1] = 2$ $\quad S[2] = -1$ $\quad S[3] = 0$ $\quad S[4] = 1$

The algorithm will choose the following path on the trellis:



$S[1] = 2$ $\quad S[2] = -1$ $\quad S[3] = 0$ $\quad S[4] = 1$

and estimates the transmitted sequence as $\hat{I}[1] = +1$, $\hat{I}[2] = -1$, $\hat{I}[3] = +1$.

(d) (i) Since the channel has two taps the length of the cyclic prefix should be at least 1. For transmitting four symbols at $I[0], \dots, I[3]$, we need to set $I[-1] = I[3]$.

(ii) The length-4 DFT of the channel response vector $(1, 1, 0, 0)$ is $(1, 1/2 - 1/2j, 0, 1/2 + 1/2j)$. Multiplying the result by $\sqrt{N} = 2$, the 4 parallel channels will have gains $(2, 1-j, 0, 1+j)$ and the noise has unit-variance in all channels. Equivalently, by dividing the noise variance by the square of channel gain, we have channels with gain 1, and noise variance $1/4$, $1/2$, $+\infty$, and $1/2$, respectively.

(iii) Using water-filling, we see that, with the constant to be $5/4$ we need to fill 1 in the first channel and $3/4$ in the second and the fourth channel (and we don't use the third channel).

(iv) With the above power allocation, the first sub-channel has capacity $\log_2(5) > 2$ and the second and fourth channels have capacity $\log_2(5/2) > 1$. Consequently, it is definitely possible to transmit 4 bits per OFDM symbol reliably and, to do so, we shall used QPSK modulation for the first sub-channel and BPSK for the second and fourth sub-channels.

SOLUTION 2.

(a) The Alamouti scheme transmits $\boldsymbol{x}[1] = [U_1, U_2]^T$ and $\boldsymbol{x}[2] = [-U_2^*, U_1^*]$

(b) Note that, by assumption, the channel remains constant in two consecutive uses. Therefore,

$$\boldsymbol{Y}[1] = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \boldsymbol{Z}[1]$$

and

$$\boldsymbol{Y}[2] = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} -U_2^* \\ U_1^* \end{bmatrix} + \boldsymbol{Z}[2]$$

or equivalently

$$\boldsymbol{Y}[2] = \begin{bmatrix} H_{12} & -H_{11} \\ H_{22} & -H_{21} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^* + \boldsymbol{Z}[2]$$

Taking the complex conjugate of the last equation we can write

$$\tilde{\boldsymbol{Y}} = \begin{bmatrix} Y_1[1] \\ Y_2[1] \\ Y_1[2]^* \\ Y_2[2]^* \end{bmatrix} = \underbrace{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \\ H_{12}^* & -H_{11}^* \\ H_{22}^* & -H_{21}^* \end{bmatrix}}_{=:\tilde{H}} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \underbrace{\begin{bmatrix} Z_1[1] \\ Z_2[1] \\ Z_1[2]^* \\ Z_2[2]^* \end{bmatrix}}_{=:\tilde{\boldsymbol{Z}}}$$

(c) The inner product between the columns of $\tilde{H}$ is

$$H_{11}H_{12}^* + H_{21}H_{22}^* - H_{12}^*H_{11} - H_{22}^*H_{21} = 0$$

The *informative* component of $\tilde{\boldsymbol{Y}}$ lies in the signal space spanned by the columns of $\tilde{H}$. Therefore, projection of $\tilde{\boldsymbol{Y}}$ onto this space gives us sufficient statistic for decision. We also note that both vectors have norm $\|H\| = \sqrt{|H_{11}|^2 + |H_{12}|^2 + |H_{21}|^2 + |H_{22}|^2}$. In particular, since the receiver knows the channel realization $H = h$, a sufficient statistic for decision would be

$$T_1 = \frac{1}{\|h\|} \begin{bmatrix} h_{11}^* & h_{21}^* & h_{12} & h_{22} \end{bmatrix} \tilde{\boldsymbol{Y}} = \|h\| U_1 + W_1$$
$$T_2 = \frac{1}{\|h\|} \begin{bmatrix} h_{12}^* & h_{22}^* & -h_{11} & -h_{21} \end{bmatrix} \tilde{\boldsymbol{Y}} = \|h\| U_2 + W_2$$

where W_1 and W_2 are independent circularly-symmetric Gaussian variables of unit variance.

(d) Given the sufficient statistic of (c), the ML decision would be

$$\hat{U}_1 = \begin{cases} +A & \text{if } \Re\{T_1\} \geq 0 \\ -A & \text{if } \Re\{T_1\} \leq 0 \end{cases}$$

and

$$\hat{U}_2 = \begin{cases} +A & \text{if } \Re\{T_2\} \geq 0 \\ -A & \text{if } \Re\{T_2\} \leq 0 \end{cases}$$

Conditioned on $H = h$, the probability of decision error is

$$P_e(h) = Q\left(2\|h\|A\right) \leq \exp(-2\|h\|^2 A^2)$$

(Note that since we only look at the real parts of T_1 and T_2 only the real parts of W_1 and W_2 which have variance $\frac{1}{2}$ influence the decision.) Taking the expectation of the above with respect to H we get

$$\overline{P_e} \lessapprox \frac{1}{(1+2A^2)^4}$$

(To compute the expectation we note that $\|H\|^2 = |H_{11}|^2 + |H_{12}|^2 + |H_{21}|^2 + |H_{22}|^2$ is the sum of independent exponentially distributed random variables with mean 1, thus $\exp(-2\|H\|^2 A^2)$ can be factored into the product of independent random variables.) This means we have a diversity of order 4 for each information bit.

Solution 3.

(a) To show that the degree distribution pair is well-formed we need to show that all coefficients are non-negative and sum up to 1. The non-negativity follows from the hint. The fact that the sum is one follows by setting $x = 1$.

(b) The rate for a degree distribution pair $(\lambda(x), \rho(x))$ is given by

$$R = 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx}.$$

Using the hint we see that the rate is $\tilde{R} = \frac{16}{21}$.

(c) Let us write down the density evolution equation for the pair $(\tilde{\lambda}, \tilde{\rho})$. We have

$$\begin{aligned}
\tilde{f}(\epsilon, x) &= \epsilon\tilde{\lambda}(1 - \tilde{\rho}(1-x)) \\
&= \epsilon(1 - (1-\sqrt{x})^5)^4 \\
&= (\sqrt{\epsilon}(1 - (1-\sqrt{x})^5)^2)^2 \\
&= (\sqrt{\epsilon}\lambda(1 - \rho(1-\sqrt{x}))^2 \\
&= f(\sqrt{\epsilon}, \sqrt{x}).
\end{aligned}$$

Hence the DE converges to 0 as long as $\sqrt{\epsilon} \leq \epsilon^*(\lambda, \rho)$.