ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 32 Advanced Digital Communications
Final Exam Jan. 24, 2017

- 3 problems, 60 + 5 bonus points, 180 minutes
- This is a closed book exam.
- Only *two* double-sided handwritten A4 pages of summary allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET

Useful facts:

- Let $x[n]$ be a signal of length N and let $X[n]$ be its DFT. Let $x \circledast y$ denote the cyclic convolution of the two signals x and y. Then we have

$$X[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[k] e^{-2\pi j \frac{kn}{N}}, \qquad x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{2\pi j \frac{kn}{N}}, \qquad x \circledast y \longleftrightarrow \sqrt{N} XY.$$

- $$Q(\alpha) \leq \exp\left(-\frac{\alpha^2}{2}\right).$$

- $$\sqrt{1-x} = 1 - \sum_{i\geq 1} c_i x^i, c_i \geq 0, \qquad \int_0^1 (1-\sqrt{1-x})^5 dx = \frac{1}{21}.$$

- All numbers, in all problems, are extremely simple. If you get complicated numbers go back to square one!

PROBLEM 1. (25 points) Consider the problem of transmitting i.i.d., zero-mean, and unit-variance data symbols $I[n]$ through the discrete-time ISI channel given by the model

$$U[n] = 2I[n] + I[n+1] + I[n-1] + V[n],$$

where $V[n]$ is a zero-mean circularly symmetric Gaussian random process with auto-correlation function

$$R_V[k] = 2\delta[k] + \delta[k-1] + \delta[k+1]$$

(a) (4 pts) Assume we wish to combat the ISI by filtering the channel output $U[n]$ through an equalizer to obtain

$$\hat{I}[n] = I[n] + \tilde{V}[n],$$

where $\tilde{V}[n]$ is the residual noise after equalization. The choices for the equalizer are zero-forcing and LMMSE filtering. Explain which filter will you pick in order for $\tilde{V}[n]$ to have the smallest variance and write down the frequency response of the equalizer.

Note. No need to compute noise variances to compare the equalizers, just think about the design objective for each of them and explain your choice.

For the rest of the problem, assume we pass the channel output $U[n]$ through a whitening filter $D_{\mathrm{W}}(z)$ to obtain a *causal* effective channel described as

$$S[n] = \sum_{k \geq 0} I[n-k] f[k] + W[n]$$

where $W[n]$ is additive white Gaussian noise of variance 1.

(b) (4 pts) Determine the whitening filter $D_{\mathrm{W}}(z)$ and, accordingly, the channel coefficients $f[k]$.

(c) (9 pts) Assume that $I[n]$ are binary symbols chosen uniformly from $\{\pm 1\}$ and we use the Viterbi algorithm to estimate the transmitted sequence $I[n]$ upon observing $S[n]$.

  (i) Draw the trellis and label the edges with *noiseless* channel output.

  (ii) Describe the metric that the Viterbi algorithm assigns to each edge and determine its estimation of the transmitted sequence, $\hat{I}[n]$, if the channel output sequence is $S[1] = 2$, $S[2] = -1$, $S[3] = 0$, and $S[4] = 1$? As always, assume $I[0] = I[4] = +1$.

(d) (8 pts) Assume now that instead of a Viterbi decoder, we use an OFDM system with symbol length $N = 4$ over the effective channel between $I[n]$ and $S[n]$ to combat the ISI.

  (i) What is the minimum length of the cyclic prefix (guard interval) that you need to use? Assuming that we transmit at 4 positions $I[0], \ldots, I[3]$, what values do we have to assign to $I[-1], \ldots$?

  (ii) Describe the N parallel channels that you get (channel gain, noise variance).

  (iii) Suppose we have a total power budget of $10/4$ per OFDM block (of length N). What is the optimal power allocation?

  (iv) Now you are asked to pick a modulation scheme for each OFDM sub-channel so that, in total, you transmit 4 bits per OFDM symbol. The choices are BPSK and QPSK. Determine, first, if it is possible to transmit this many bits per OFDM symbol reliably and, if yes, describe which modulation you would use for each sub-channel.

Problem 2. (20 points) Consider a communication system with two transmit and two receive antennas described by the model

$$\boldsymbol{Y}[n] = H[n]\boldsymbol{x}[n] + \boldsymbol{Z}[n]$$

where $\boldsymbol{x}[n] = \begin{bmatrix} x_1[n] & x_2[n] \end{bmatrix}^T$ is the sent signal at time n by the two transmit antennas, $\boldsymbol{Y}[n] = \begin{bmatrix} Y_1[n] & Y_2[n] \end{bmatrix}^T$ is the received signals of two receive antennas (at time n), $\boldsymbol{Z}[n] \sim \mathcal{CN}(\mathbf{0}, I_2)$ is a circularly-symmetric Gaussian noise, and

$$H[n] = \begin{bmatrix} H_{11}[n] & H_{12}[n] \\ H_{21}[n] & H_{22}[n] \end{bmatrix}$$

is the matrix of channel coefficients which are assumed to be independent circularly-symmetric Gaussian with unit variance. Assume the channel is block time-invariant for two consecutive uses and suppose we wish to transmit *two* information symbols U_1 and U_2 via *two* channel uses using the Alamouti scheme. Assume also that the information symbols are $\{-A, A\}$-valued.

(a) (5 pts) Describe the signals that the antennas transmit in two time slots, $\boldsymbol{x}[1]$ and $\boldsymbol{x}[2]$.

(b) (5 pts) Note that after two consecutive transmissions the receiver has a four-dimensional observation (two samples per antenna) and has to estimate U_1 and U_2. Transform the decision problem to that of estimating (U_1, U_2) upon observing

$$\tilde{\boldsymbol{Y}} = \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \\ \tilde{Y}_3 \\ \tilde{Y}_4 \end{bmatrix} = \tilde{H} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \tilde{\boldsymbol{Z}}$$

where $\tilde{H} \in \mathbb{C}^{4\times 2}$ and $\tilde{\boldsymbol{Z}} \sim \mathcal{CN}(0, I_4)$. Write down explicitly $\tilde{\boldsymbol{Y}}$ and $\tilde{H}$.

(c) (5 pts) Show that the columns of $\tilde{H}$ are orthogonal. Using this show that the projection of $\tilde{\boldsymbol{Y}}$ on the normalized column vectors of $\tilde{H}$ gives sufficient statistics for detection of (U_1, U_2).

(d) (5 pts) Write down the ML decision rule and find an upper bound on average probability of error. Find the diversity gain *for each information bit* with this scheme.

Problem 3. (20 points) Consider a regular $(3, 6)$ LDPC code and transmission over the BEC. We saw in the course that it has rate $R = 1/2$ and a threshold of around $\epsilon^*(\lambda, \rho) = 0.4294$.

To recall, a compact way of analysing this case is to define $\lambda(x) = x^2$ and $\rho(x) = x^5$ (these two encode the degree distribution) and let $f(\epsilon, x) = \epsilon\lambda(1 - \rho(1 - x))$ be the density evolution equation, where we start with $x_0 = \epsilon$ and $x_l = f(\epsilon, x_{l-1})$, $l \geq 1$.

Saying that ϵ^* is the threshold means that for any $0 \leq \epsilon < \epsilon^*$ the sequence x_l converges to 0.

Consider now the irregular code with $\tilde{\lambda}(x) = x^4$ and $\tilde{\rho}(x) = (1 - \sqrt{1 - x})^5$.

(a) (6 pts) Show that $\tilde{\rho}(x)$ is a well-defined degree distribution.

(b) (6 pts) Determine the rate of $(\tilde{\lambda}, \tilde{\rho})$.

(c) (8 pts) Show that the threshold of $(\tilde{\lambda}, \tilde{\rho})$ is equal to

$$\epsilon^*(\tilde{\lambda}, \tilde{\rho}) = (\epsilon^*(\lambda, \rho))^2.$$

Hint. For the last part no computation is required. Just a little thinking. Start by writing down the density evolution equations and try to relate it to the density evolution equations of the original case.