Advanced Digital Communications

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Part I

Review of Signal Processing and Detection

Chapter 1

Overview

1.1 Digital data transmission

Most of us have used communication devices, either by talking on a telephone, or browsing the internet on a computer. This course is about the mechanisms that allows such communications to occur. The focus of this class is on how "bits" are transmitted through a "communication" channel. The overall communication system is illustrated in Figure 1.1

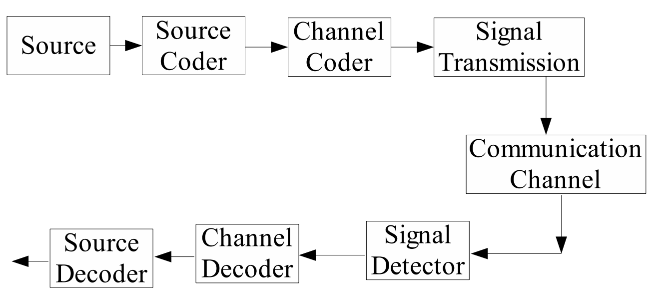


Figure 1.1: Communication block diagram.

1.2 Communication system blocks

Communication Channel: A communication channel provides a way to communicate at large distances. But there are external signals or "noise" that effects transmission. Also 'channel' might behave differently to different input signals. A main focus of the course is to understand signal processing techniques to enable digital transmission over such channels. Examples of such communication channels include: telephone lines, cable TV lines, cell-phones, satellite networks, etc. In order to study these problems precisely, communication channels are often modelled mathematically as illustrated in Figure 1.2.

Source, Source Coder, Applications: The main reason to communicate is to be able to talk, listen to music, watch a video, look at content over the internet, etc. For each of these cases the "signal"

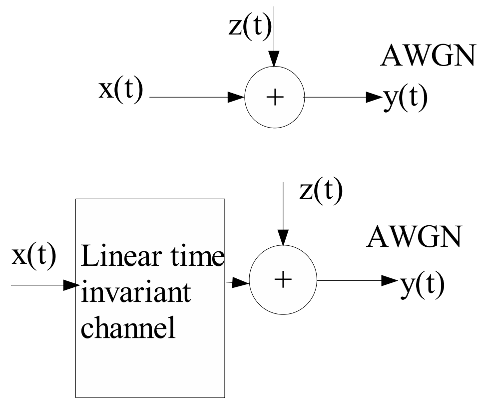


Figure 1.2: Models for communication channels.

respectively voice, music, video, graphics has to be converted into a stream of bits. Such a device is called a quantizer and a simple scalar quantizer is illustrated in Figure 1.3. There exists many quantization methods which convert and compress the original signal into bits. You might have come across methods like PCM, vector quantization, etc.

Channel coder: A channel coding scheme adds redundancy to protect against errors introduced by the noisy channel. For example a binary symmetric channel (illustrated in Figure 1.4) flips bits randomly and an error correcting code attempts to communicate reliably despite them.

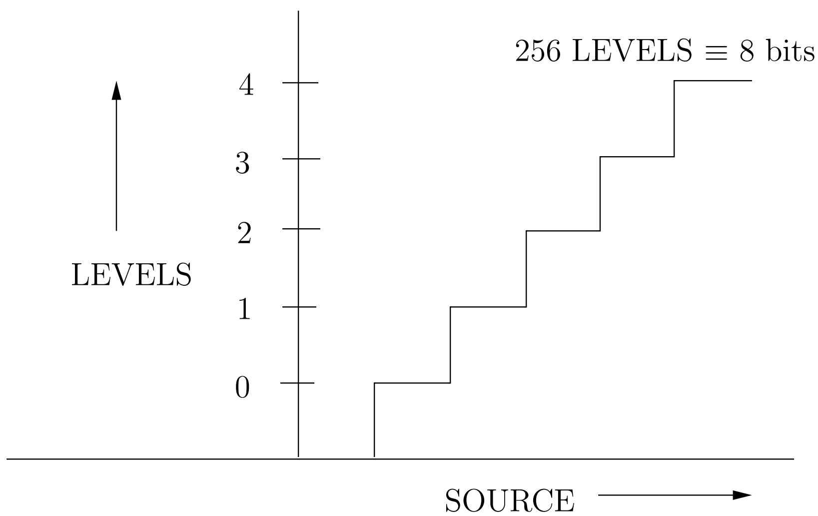


Figure 1.3: Source coder or quantizer.

Signal transmission: Converts "bits" into signals suitable for communication channel which is typically analog. Thus message sets are converted into waveforms to be sent over the communication channel.

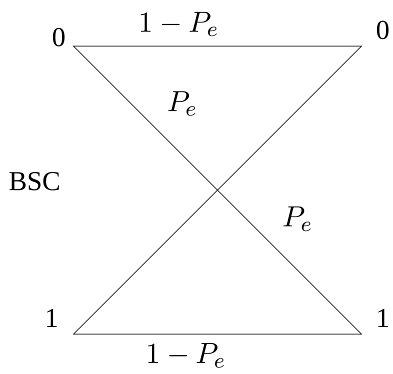


Figure 1.4: Binary symmetric channel.

This is called modulation or signal transmission. One of the main focuses of the class.

Signal detection: Based on noisy received signal, receiver decides which message was sent. This procedure called "signal detection" depends on the signal transmission methods as well as the communication channel. Optimum detector minimizes the probability of an erroneous receiver decision. Many signal detection techniques are discussed as a part of the main theme of the class.

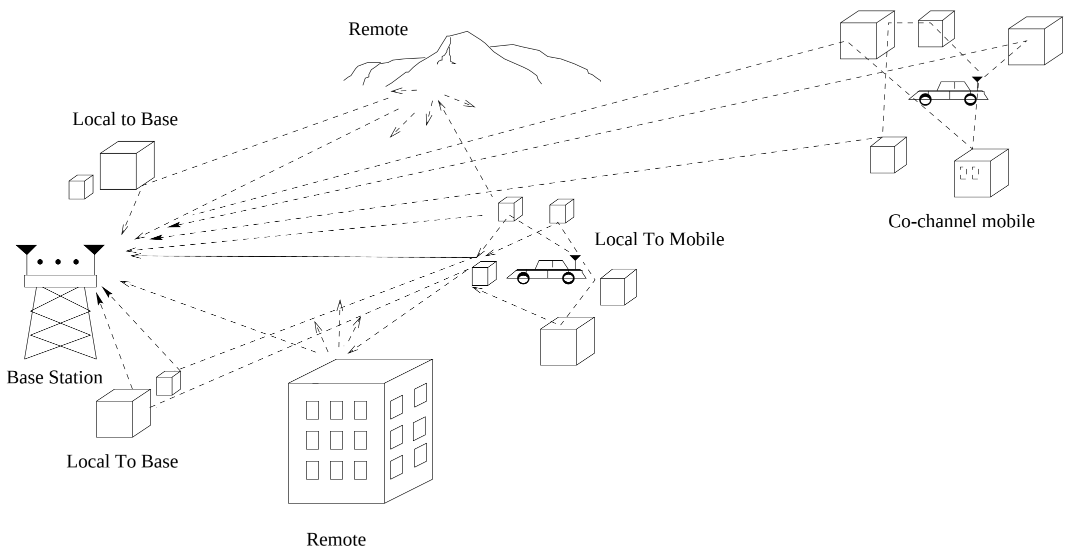


Figure 1.5: Multiuser wireless environment.

Multiuser networks: Multiuser networks arise when many users share the same communication channel. This naturally occurs in wireless networks as shown in Figure 1.5. There are many different forms of multiuser networks as shown in Figures 1.6, 1.7 and 1.8.

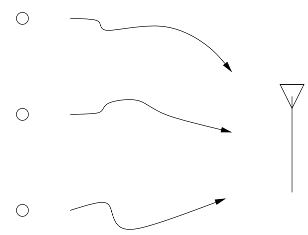


Figure 1.6: Multiple Access Channel (MAC).

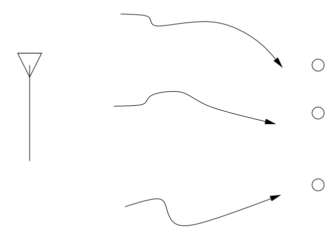


Figure 1.7: Broadcast Channel (BC).

1.3 Goals of this class

- Understand basic techniques of signal transmission and detection.
- Communication over frequency selective or inter-symbol interference (ISI) channels.
- Reduced complexity (sub-optimal) detection for ISI channels and their performances.
- Multiuser networks.
- Wireless communication - rudimentary exposition.
- Connection to information theory.

Complementary classes

- Source coding/quantization (ref.: Gersho & Gray, Jayant & Noll)
- Channel coding (Modern Coding theory, Urbanke & Richardson, Error correcting codes, Blahut)
- Information theory (Cover & Thomas)

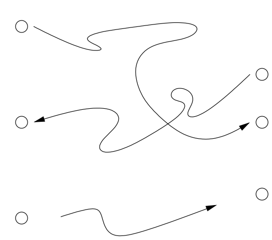


Figure 1.8: Adhoc network.

1.4 Class organization

These are the topics covered in the class.

- Digital communication & transmission
- Signal transmission and modulation
- Hypothesis testing & signal detection
- Inter-symbol interference channel - transmission & detection
- Wireless channel models: fading channel
- Detection for fading channels and the tool of diversity
- Multiuser communication - TDMA, CDMA
- Multiuser detection
- Connection to information theory

1.5 Lessons from class

These are the skills that you should know at the end of the class.

- Basic understanding of optimal detection
- Ability to design transmission & detection schemes in inter-symbol interference channels
- Rudimentary understanding of wireless channels
- Understanding wireless receivers and notion of diversity
- Ability to design multiuser detectors
- Connect the communication blocks together with information theory

Chapter 2

Signals and Detection

2.1 Data Modulation and Demodulation

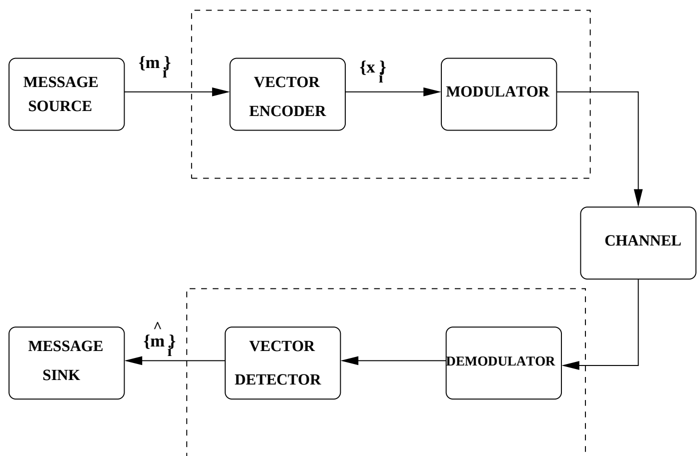


Figure 2.1: Block model for the modulation and demodulation procedures.

In data modulation we convert information bits into waveforms or signals that are suitable for transmission over a communication channel. The detection problem is reversing the modulation, *i.e.*, finding which bits were transmitted over the noisy channel.

Example 2.1.1. *(see Figure 2.2) Binary phase shift keying. Since DC does not go through channel, this implies that 0V, and 1V, mapping for binary bits will not work. Use:*

$$x_0(t) = \cos(2\pi 150t), \;\; x_1(t) = -\cos(2\pi 150t).$$

Detection: *Detect +1 or -1 at the output.*
Caveat: *This is for single transmission. For successive transmissions, stay tuned!*

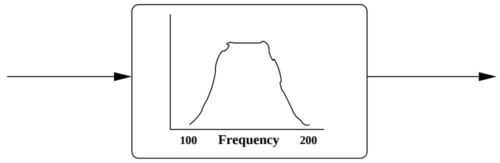


Figure 2.2: The channel in example 1.

2.1.1 Mapping of vectors to waveforms

Consider set of real-valued functions $\{f(t)\}, t \in [0, T]$ such that

$$\int_0^T f^2(t)dt < \infty$$

This is called a Hilbert space of continuous functions, *i.e.,* $\mathcal{L}_2[0, T]$.

Inner product

$$< f, g > = \int_0^T f(t)g(t)dt.$$

Basis functions: A class of functions can be expressed in terms of basis functions $\{\phi_n(t)\}$ as

$$x(t) = \sum_{n=1}^{N} x_n \phi_n(t), \tag{2.1}$$

where $< \phi_n, \phi_m >= \delta_{n-m}$. The waveform carries the information through the communication channel. Relationship in (2.1) implies a mapping $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ to $x(t)$.

Definition 2.1.1. ***Signal Constellation*** *The set of M vectors $\{\mathbf{x}_i\}, i = 0, \ldots, M-1$ is called the signal constellation.*

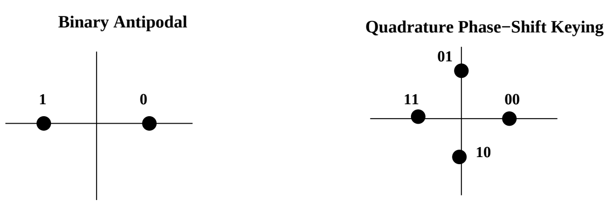


Figure 2.3: Example of signal constellations.

The mapping in (2.1) enables mapping of points in $\mathcal{L}_2[0, T]$ with properties in $\mathbb{R}^N$. If $x_1(t)$ and $x_2(t)$ are waveforms and their corresponding basis representation are $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively, then,

$$< x_1, x_2 >=< \mathbf{x}_1, \mathbf{x}_2 >$$

where the left side of the equation is $< x_1, x_2 >= \int_0^T x_1(t)x_2(t)dt$ and the right side is $< \mathbf{x}_1, \mathbf{x}_2 >= \sum_{i=1}^{N} \mathbf{x}_1(i)\mathbf{x}_2(i)$.

Examples of signal constellations: Binary antipodal, QPSK (Quadrature Phase Shift Keying).

Vector Mapper: Mapping of binary vector into one of the signal points. Mapping is not arbitrary, clever choices lead to better performance over noisy channels.
In some channels it is suitable to label points that are "close" in Euclidean distance to map to being "close" in Hamming distance. Examples of two alternate labelling schemes are illustrated in Figure 2.4.

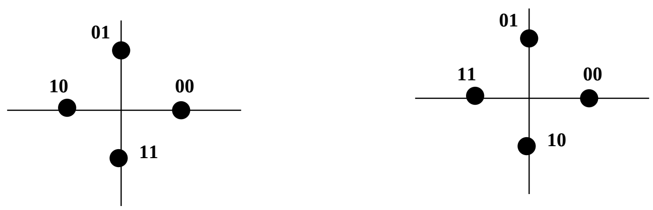


Figure 2.4: A vector mapper.

Modulator: Implements the basis expansion of (2.1).

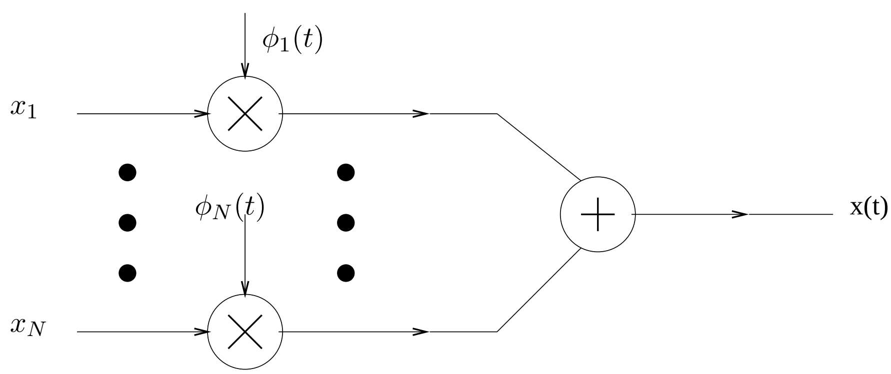


Figure 2.5: Modulator implementing the basis expansion.

Signal Set: Set of modulated waveforms $\{x_i(t)\}, i = 0, \ldots, M-1$ corresponding to the signal constellation $\mathbf{x}_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,N} \end{bmatrix} \in \mathbb{R}^N$.

Definition 2.1.2. ***Average Energy:***

$$\mathcal{E}_x = \mathbb{E}[||\mathbf{x}||^2] = \sum_{i=0}^{M-1} ||\mathbf{x}_i||^2 p_{\mathbf{X}}(i)$$

where $p_{\mathbf{X}}(i)$ is the probability of choosing $\mathbf{x}_i$.

The probability $p_x(i)$ depends on,

- Underlying probability distribution of bits in message source.
- The vector mapper.

Definition 2.1.3. ***Average power:*** $P_x = \frac{\mathcal{E}_x}{T}$ *(energy per unit time)*

Example 2.1.2. *Consider a 16 QAM constellation with basis functions:*

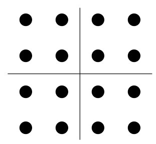


Figure 2.6: 16 QAM constellation.

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos \frac{\pi t}{T}, \ \ \phi_2(t) = \sqrt{\frac{2}{T}} \sin \frac{\pi t}{T}$$

For $\frac{1}{T} = 2400Hz$*, we get a rate of* $\log(16) \times 2400 = 9.6kb/s$*.*

Gram-Schmidt procedure allows choice of minimal basis to represent $\{x_i(t)\}$ signal sets. More on this during the review/exercise sessions.

2.1.2 Demodulation

The demodulation takes the continuous time waveforms and extracts the discrete version. Given the basis expansion of (2.1), the demodulation extracts the coefficients of the expansion by projecting the signal onto its basis as shown below.

$$\begin{aligned} x(t) &= \sum_{k=1}^{N} x_k \phi_k(t) \qquad (2.2) \\ \Longrightarrow \int_0^T x(t)\phi_n(t)dt &= \int_0^T \sum_{k=1}^{N} x_k \phi_k(t)\phi_n(t)dt \\ = \sum_{k=1}^{N} x_k \int_0^T \phi_k(t)\phi_n(t)dt &= \sum_{k=1}^{N} x_k \delta_{k-n} = x_n \end{aligned}$$

Therefore in the noiseless case, *demodulation* is just recovering the coefficients of the basis functions.

Definition 2.1.4. ***Matched Filter:*** *The matched filter operation is equivalent to the recovery of the coefficients of the basis expansion since we can write as an equation:* $\int_0^T x(t)\phi_n(t)dt == x(t) * \phi_n(T - t)|_{t=T} = x(t) * \phi_n(-t)|_{t=0}$.

Therefore, the basis coefficients recovery can be interpreted as a filtering operation.

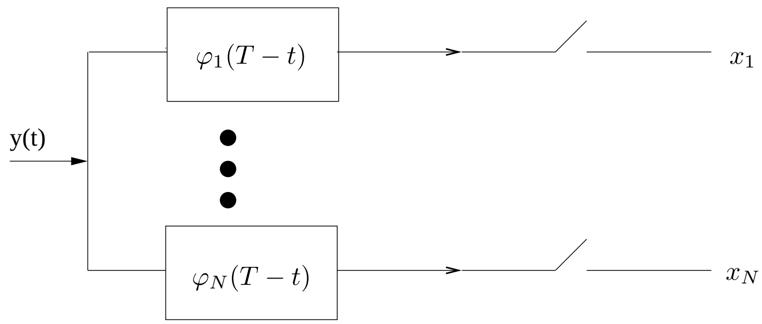


Figure 2.7: Matched filter demodulator.

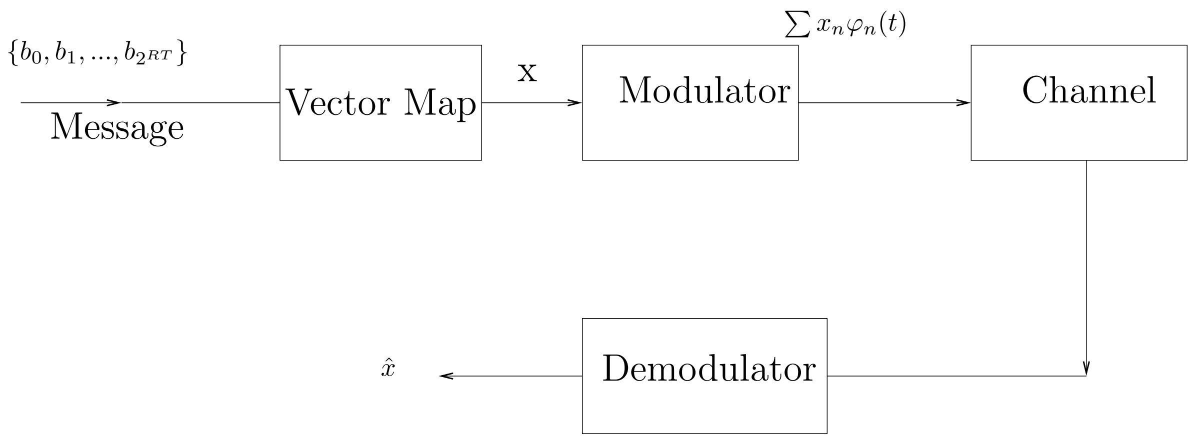


Figure 2.8: Modulation and demodulation set-up as discussed up to know.

2.2 Data detection

We assume that the demodulator captures the "essential" information about $\mathbf{x}$ from $y(t)$. This notion of "essential" information will be explored in more depth later.
In discrete domain:

$$P_Y(y) = \sum_{i=0}^{M-1} p_{Y|X}(y|i) p_X(i)$$

This is illustrated in Figure 2.9 showing the equivalent discrete channel.

Example 2.2.1. *Consider the Additive White Gaussian Noise Channel (AWGN). Here* $\mathbf{y} = \mathbf{x} + \mathbf{z}$*, and*

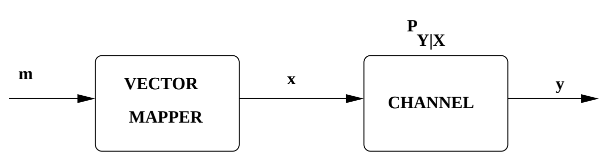


Figure 2.9: Equivalent discrete channel.

hence $p_{Y|X}(y|x) = p_Z(y-x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}}$.

2.2.1 Criteria for detection

Detection is guessing input $\mathbf{x}$ given the noisy output $\mathbf{y}$. This is expressed as a function $\hat{m} = H(y)$.
If $\mathcal{M} = m$ was the message sent, then

$$\text{Probability of error} = P_e \stackrel{def}{=} \text{Prob}(\hat{m} \neq m).$$

Definition 2.2.1. ***Optimum detector:*** *Minimizes error probability over all detectors. The probability of observing Y=y if the message m_i was sent is,*

$$p(\mathbf{Y} = \mathbf{y} \mid \mathcal{M} = m_i) = p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid i)$$

Decision Rule: $H : \mathbf{Y} \to \mathcal{M}$ is a function which takes input $\mathbf{y}$ and outputs a guess on the transmitted message. Now,

$$\mathbb{P}(H(\mathbf{Y})\ \texttt{is correct}) = \int_{\mathbf{y}} \mathbb{P}[H(\mathbf{y})\ \texttt{is correct} \mid \mathbf{Y} = \mathbf{y}] p_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} \tag{2.3}$$

Now $H(\mathbf{y})$ is a deterministic function of ybf which divides the space $\mathbb{R}^N$ into M regions corresponding to each of the possible hypotheses. Let us define these *decision regions* by

$$\Gamma_i = \{\mathbf{y} : H(\mathbf{y}) = m_i\}, \ \ i = 0, \ldots, M-1. \tag{2.4}$$

Therefore, we can write (2.3) as,

$$\begin{aligned}
\mathbb{P}(H(\mathbf{Y})\ \texttt{is correct}) &= \sum_{j=0}^{M-1} \mathbb{P}(\mathbf{x} = \mathbf{x}_j) \mathbb{P}(H(\cdot) = m_j | \mathbf{x} = \mathbf{x}_j) &(2.5)\\
&= \sum_{j=0}^{M-1} \mathbb{P}(\mathbf{x} = \mathbf{x}_j) \int_{\mathbf{y} \in \Gamma_j} \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_j}) d\mathbf{y}\\
&= \sum_{j=0}^{M-1} \mathbb{P}(\mathbf{x} = \mathbf{x}_j) \int_{\mathbf{y}} 1\!\!1_{\{\mathbf{y} \in \Gamma_j\}} \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_j}) d\mathbf{y}\\
&= \int_{\mathbf{y}} \left[\sum_{j=0}^{M-1} \mathbb{P}(\mathbf{x} = \mathbf{x}_j) 1\!\!1_{\{\mathbf{y} \in \Gamma_j\}} \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_j}) \right] d\mathbf{y}\\
&\stackrel{(a)}{\leq} \int_{\mathbf{y}} \max_{j=0,\ldots M-1} \left[\mathbb{P}(\mathbf{x} = \mathbf{x}_j) \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_j}) \right] d\mathbf{y}\\
&= \int_{\mathbf{y}} \{\max_j \mathbb{P}_{\mathbf{X}|\mathbf{Y}}[\mathbf{X} = \mathbf{x}_j \mid \mathbf{y}]\} p_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y}\\
&= \mathbb{P}(H_{MAP}(\mathbf{Y})\ \texttt{is correct}) &(2.6)
\end{aligned}$$

where $1\!\!1_{\{\mathbf{y} \in \Gamma_j\}}$ is the indicator function which is 1 if $\mathbf{y} \in \Gamma_j$ and 0 otherwise. Now (a) follows because $H(\cdot)$ is a deterministic rule, and hence $1\!\!1_{\{\mathbf{y} \in \Gamma_j\}}$ can be 1 for only exactly one value of j for each $\mathbf{y}$. Therefore, the optimal decision regions are:

$$\Gamma_i^{\text{MAP}} = \{\mathbf{y} : i = \arg \max_{j=0,\ldots,M-1} \mathbb{P}_{\mathbf{X}|\mathbf{Y}}[\mathbf{X} = \mathbf{x}_j \mid \mathbf{y}]\}, \ \ i = 0, \ldots, M-1. \tag{2.7}$$

Implication: The decision rule

$$H_{MAP}(\mathbf{y}) = \arg\max_i \mathbb{P}_{\mathbf{X}|\mathbf{Y}}[\mathbf{X} = \mathbf{x}_i \mid \mathbf{y}]$$

maximizes probability of being correct, *i.e.,* minimizes error probability. Therefore, this is the *optimal* decision rule. This is called the **Maximum-a-posteriori** (MAP) decision rule.

Notes:

- MAP detector needs knowledge of the priors $\mathtt{p}_{\mathbf{X}}(\mathbf{x})$.
- It can be simplified as follows:

$$\mathtt{p}_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}_i \mid \mathbf{y}) = \frac{\mathtt{p}_{\mathbf{Y}|\mathbf{X}}[\mathbf{y} \mid \mathbf{x}_i]\mathtt{p}_{\mathbf{X}}(\mathbf{x}_i)}{\mathtt{p}_{\mathbf{Y}}(\mathbf{y})} \equiv \mathtt{p}_{\mathbf{Y}|\mathbf{X}}[\mathbf{y} \mid \mathbf{x}_i]\mathtt{p}_{\mathbf{X}}(\mathbf{x}_i)$$

  since $\mathtt{p}_{\mathbf{Y}}(\mathbf{y})$ is common to all hypotheses. Therefore the MAP decision rule is equivalently written as:

$$H_{MAP}(\mathbf{y}) = \arg\max_i \mathtt{p}_{\mathbf{Y}|\mathbf{X}}[\mathbf{y} \mid \mathbf{x}_i]\mathtt{p}_{\mathbf{X}}(\mathbf{x}_i)$$

An alternate proof for MAP decoding rule (binary hypothesis)

Let Γ_0, Γ_1 be the decision regions for the messages m_0, m_1 as given in (2.4).

For $\pi_0 = P_{\mathbf{X}}(x_0)$ and $\pi_1 = P_{\mathbf{X}}(x_1)$

$$\begin{aligned}
\mathbb{P}[\text{error}] = \mathbb{P}[H(\mathbf{y})\texttt{is wrong}] &= \pi_0\mathbb{P}[\mathbf{y} \in \Gamma_1 \mid H_0] + \pi_1\mathbb{P}[\mathbf{y} \in \Gamma_0 \mid H_1] \\
&= \pi_0 \int_{\Gamma_1} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_0})d\mathbf{y} + \pi_1 \int_{\Gamma_0} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_1})d\mathbf{y} \\
&= \pi_0 \int_{\Gamma_1} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_0})d\mathbf{y} + \pi_1 \left[1 - \int_{\Gamma_1} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_1})d\mathbf{y}\right] \\
&= \pi_1 + \int_{\Gamma_1} \left[\pi_0 P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_0}) - \pi_1 P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_1})\right] d\mathbf{y} \\
&= \pi_1 + \int_{\mathbb{R}^N} \underbrace{\mathbb{1}_{\{\mathbf{y}\in\Gamma_1\}} \left[\pi_0 P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_0}) - \pi_1 P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x_1})\right] d\mathbf{y}}_{to\ make\ this\ term\ the\ smallest,\ collect\ all\ the\ negative\ area}
\end{aligned} \tag{2.8}$$

Therefore, in order to make the error probability smallest, we choose on $\mathbf{y} \in \Gamma_1$ if

$$\pi_0 P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid x_0) \quad < \quad \pi_1 P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid x_1)$$

That is, Γ_1 is defined as,

$$\frac{P_{\mathbf{X}}(x_0)P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid x_0)}{P_{\mathbf{Y}}(\mathbf{y})} \quad < \quad \frac{P_{\mathbf{X}}(x_1)P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid x_1)}{P_{\mathbf{Y}}(\mathbf{y})}$$

or $\mathbf{y} \in \Gamma_1$, if,

$$P_{\mathbf{X}|\mathbf{Y}}(x_0 \mid \mathbf{y}) \quad < \quad P_{\mathbf{X}|\mathbf{Y}}(x_1 \mid \mathbf{y})$$

i.e., the MAP rule!

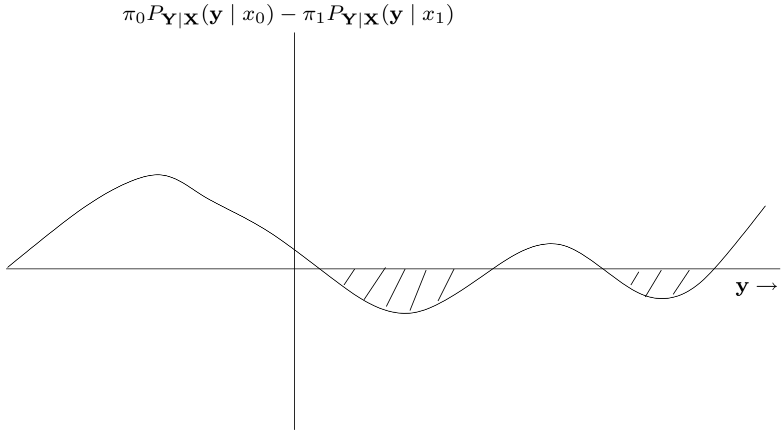


Figure 2.10: Functional dependence of integrand in (2.8).

Maximum Likelihood detector: If the priors are assumed uniform, *i.e.*, $\mathbf{p_X}(\mathbf{x}_i) = \frac{1}{M}$ then the MAP rule becomes,

$$H_{ML}(\mathbf{y}) = \arg\max_i \mathbf{p}_{\mathbf{Y}|\mathbf{X}}[\mathbf{y} \mid \mathbf{x}_i]$$

which is called the Maximum-Likelihood rule. This because it chooses the message that most likely caused the observation (ignoring how likely the message itself was). This decision rule is clearly inferior to MAP for non-uniform priors.

Question: Suppose the prior probabilities were unknown, is there a "robust" detection scheme?
One can think of this as a "game" where nature chooses the prior distribution and the detection rule is under our control.

Theorem 2.2.1. *The ML detector minimizes the maximum possible average error probability when the input distribution is unknown and if the conditional probability of error* $\mathfrak{p}[H_{ML}(\mathbf{y})\, is\, incorrect \mid \mathcal{M} = m_i]$ *is independent of i.*

Proof: Assume that $P_{e,ML|m=m_i}$ is independent of i.
Let

$$P_{e,ML|m=m_i} = P_e^{ML}(i) \stackrel{def}{=} \mathbb{P}^{ML}$$

Hence

$$P_{e,ML}(P_x) = \sum_{i=0}^{M-1} P_{\mathbf{X}}(i) P_{e,ML|m=m_i} = \mathbb{P}^{ML} \tag{2.9}$$

Therefore

$$\underset{P_{\mathbf{X}}}{max} P_{e,ML} = \underset{P_{\mathbf{X}}}{max} \sum_{i=0}^{M-1} P_{\mathbf{X}}(i) P_{e,ML|m=m_i} = \mathbb{P}^{ML}$$

For any hypothesis test H,

$$\begin{aligned}
\underset{P_{\mathbf{X}}}{max} P_{e,H} &= \underset{P_{\mathbf{X}}}{max} \sum_{i=0}^{M-1} P_{\mathrm{X}}(i) P_{e,H|m=m_i} \\
&\overset{(a)}{\geq} \sum_{i=0}^{M-1} \frac{1}{M} P_{e,H|m=m_i} \\
&\overset{(b)}{\geq} \sum_{i=0}^{M-1} \frac{1}{M} P_{e,ML|m=m_i} = P_{e,ML}
\end{aligned}$$

where (a) is because a particular choice of $P_{\mathbf{X}}$ can only be smaller than the maximum. And (b) is because the ML decoder is optimal for the uniform prior.
Thus,

$$\underset{P_{\mathbf{X}}}{max} P_{e,H} \geq P_{e,ML} = \mathbb{P}^{ML},$$

since due to (2.9)

$$P_{e,ML} = \mathbb{P}^{ML}, \ \forall P_x.$$

□

Interpretation: ML decoding is not just a simplification of the MAP rule, but also has some canonical "robustness" properties for detection under uncertainty of priors, if the regularity condition of theorem 2.2.1 is satisfied. We will explore this further in Section 2.2.2.

Example 2.2.2. ***The AWGN channel:***
Let us assume the following,

$$\mathbf{y} = \mathbf{x}_i + \mathbf{z}\,,$$

where

$$\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}), \quad \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^N$$

Hence

$$p_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{\frac{-||\mathbf{z}||^2}{2\sigma^2}}$$

giving

$$p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}) = p_{\mathbf{Z}}(\mathbf{y} - \mathbf{x})$$

MAP decision rule for AWGN channel

$p_{\mathbf{Y}|\mathbf{X}}[\mathbf{y} \mid \mathbf{x}_i] = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{\frac{-||\mathbf{y}-\mathbf{x}_i||^2}{2\sigma^2}}$

$p_{\mathbf{X}|\mathbf{Y}}[\mathbf{X} = \mathbf{x}_i \mid \mathbf{y}] = \frac{p_{\mathbf{Y}|\mathbf{X}}[\mathbf{y}|\mathbf{x}_i] p_{\mathbf{X}}(\mathbf{x}_i)}{p_{\mathbf{Y}}(\mathbf{y})}$

Therefore the MAP decision rule is:

$$\begin{aligned}
H_{MAP}(\mathbf{y}) &= \arg\max_i \left\{ p_{\mathbf{X}|\mathbf{Y}}[\mathbf{X} = \mathbf{x}_i \mid \mathbf{y}] \right\} = \arg\max_i \left\{ p_{\mathbf{Y}|\mathbf{X}}[\mathbf{y} \mid \mathbf{x}_i] p_{\mathbf{X}}(\mathbf{x}_i) \right\} \\
&= \arg\max_i \left\{ p_{\mathbf{X}}(\mathbf{x}_i) \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{\frac{-||\mathbf{y}-\mathbf{x}_i||^2}{2\sigma^2}} \right\} \\
&= \arg\max_i \left\{ \log[p_{\mathbf{X}}(\mathbf{x}_i)] - \frac{||\mathbf{y} - \mathbf{x}_i||^2}{2\sigma^2} \right\} \\
&= \arg\min_i \left\{ \frac{||\mathbf{y} - \mathbf{x}_i||^2}{2\sigma^2} - \log[p_{\mathbf{X}}(\mathbf{x}_i)] \right\}
\end{aligned}$$

ML decision rule for AWGN channels

$$\begin{aligned}
H_{ML}(\mathbf{y}) &= \arg\max_i \left\{ p_{\mathbf{Y}|\mathbf{X}}[\mathbf{y} \mid \mathbf{X} = \mathbf{x}_i] \right\} \\
&= \arg\max_i \left\{ \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{\frac{-||\mathbf{y}-\mathbf{x}_i||^2}{2\sigma^2}} \right\} \\
&= \arg\max_i \left\{ -\frac{||\mathbf{y}-\mathbf{x}_i||^2}{2\sigma^2} \right\} \\
&= \arg\min_i \left\{ \frac{||\mathbf{y}-\mathbf{x}_i||^2}{2\sigma^2} \right\}
\end{aligned}$$

Interpretation: *The maximum likelihood decision rule selects the message that is closest in Euclidean distance to received signal.*

Observation: In both MAP and ML decision rules, one does not need $\mathbf{y}$, but just the functions, $\| \mathbf{y}-\mathbf{x}_i \|^2$, $i \in 0, ..., M-1$ in order to evaluate the decision rule. Therefore, there is *no loss* of information if we retain scalars, $\{\| \mathbf{y}-\mathbf{x}_i \|^2\}$ instead of $\mathbf{y}$. In this case, it is moot, but in continuous detection, this reduction is important. Such a function that retains the "essential" information about the parameter of interest is called a *sufficient statistic*.

2.2.2 Minmax decoding rule

The MAP decoding rule needs the knowledge of the prior distribution $\{P_{\mathbf{X}}(\mathbf{x} = x_i)\}$. If the prior is unknown we develop a criterion which is "robust" to the prior distribution. Consider the criterion used by nature

$$\max_{P_{\mathbf{X}}} \min_H P_{e,H}(p_x)$$

and the criterion used by the designer

$$\min_H \max_{P_{\mathbf{X}}} P_{e,H}(p_x)$$

where $P_{e,H}(p_x)$ is the error probability of decision rule H, *i.e.,*

$$\begin{array}{cc}
\mathbb{P}[H(\mathbf{y}) \textit{ is incorrect}] & \textit{explicitly depends on } P_{\mathbf{X}}(\mathbf{x}) \\
\| & \\
P_{e,H}(p_X) &
\end{array}$$

For the binary case,

$$\begin{aligned}
P_{e,H}(p_X) &= \pi_0 \underbrace{\mathbb{P}[\mathbf{y} \in \Gamma_1 \mid x_0]}_{\textit{does not depend on } \pi_1, \pi_0} + \pi_1 \underbrace{\mathbb{P}[\mathbf{y} \in \Gamma_0 \mid x_1]}_{\textit{does not depend on } \pi_1, \pi_0} \\
&= \pi_0 \int_{\Gamma_1} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid x_0) d\mathbf{y} + (1-\pi_0) \int_{\Gamma_0} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid x_1) d\mathbf{y}
\end{aligned}$$

Thus for a given decision rule H which does **not** depend on p_x, $P_{e,H}(p_X)$ is a **linear** function of $P_X(x)$. A "robust" detection criterion is when we want to

$$\min_H \max_{\pi_0} P_{e,H}(\pi_0).$$

Clearly for a *given* decision rule H,

$$\max_{\pi_0} P_{e,H}(\pi_0) = \max\{\mathbb{P}[\mathbf{y} \in \Gamma_1 \mid H_0],\ \mathbb{P}[\mathbf{y} \in \Gamma_0 \mid H_1]\} \tag{2.10}$$

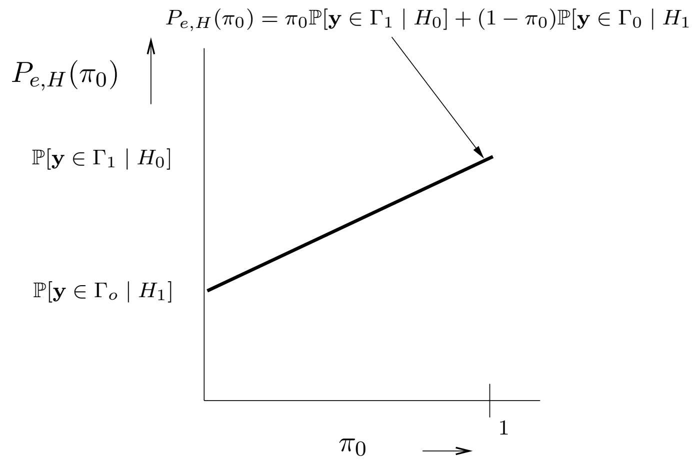


Figure 2.11: $P_{e,ML}(\pi_0)$ as a function of the prior π_0.

Now let us look at the MAP rule for every choice of π_0.
Let $V(\pi_0) = P_e^{MAP}(\pi_0)$ *i.e.,* the error probability of the MAP decoding rule as a function of $P_{\mathbf{X}}(\mathbf{x})$ (or π_0).

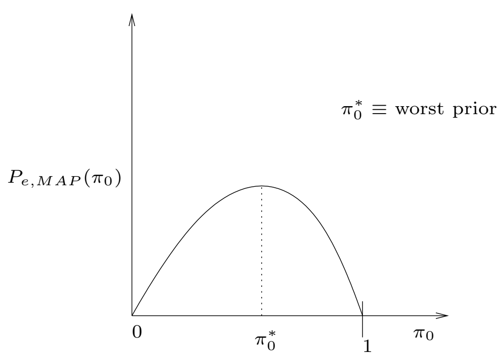


Figure 2.12: The average error probability $P_{e,ML}(\pi_0)$ of the MAP rule as a function of the prior π_0.

Since the MAP decoding rule does depend on $P_{\mathbf{X}}(\mathbf{x})$, the error probability is no longer a linear function and is actually concave (see Figure 2.12, and HW problem). Such a concave function has a unique maximum value and if it is strictly concave has a unique maximizer π_0^*. This value $V(\pi_0^*)$ is the largest average error probability for the MAP detector and π_0^* is the worst prior for the MAP detector.

Now, for any decision rule that does *not* depend on $P_{\mathbf{X}}(\mathbf{x})$, $P_{e,H}(p_x)$ is a linear function of π_0 (for the binary case) and this is illustrated in Figure 2.11. Since $P_{e,H}(p_x) \geq P_{e,MAP}(p_x)$ for each p_x. The line always lies above the curve $V(\pi_0)$. The best we could do is to make it tangential to $V(\pi_0)$ for some $\tilde{\pi}_0$, as shown in Figure 2.13. This means that such a decision rule is the MAP decoding rule designed for prior $\tilde{\pi}_0$. If we want the $\max\limits_{P_{\mathbf{X}}} P_{e,H}(p_x)$ to be the smallest it is clear that we want $\tilde{\pi}_0 = \pi_0^*$, *i.e.,* design the robust detection rule as the MAP rule for π_0^*. Since π_0^* is the worst prior for the MAP rule, this is the best one could hope for. Since the tangent to $V(\pi_0)$ at π_0^* has slope 0, such a detection rule has the property that $P_{e,H}(\pi_0)$ is independent of π_0.

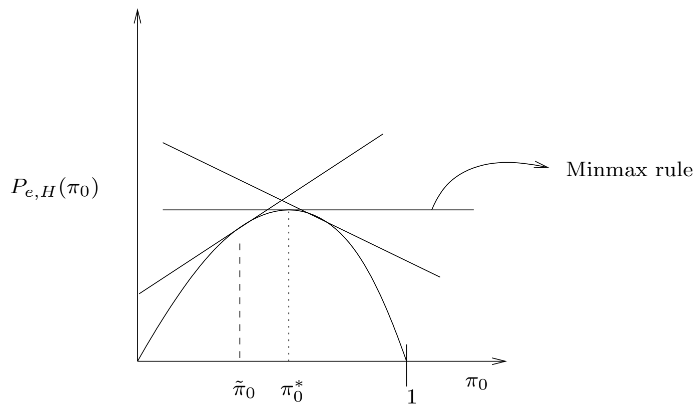


Figure 2.13: $P_{e.H}(\pi_0)$ and $P_{e,MAP}(\pi_0)$ as a function of prior π_0.

Therefore, for the minmax rule H^*, we would have

$$\mathbb{P}_{H^*}[\mathbf{y} \in \Gamma_1 \mid H_0] = \mathbb{P}_{H^*}[\mathbf{y} \in \Gamma_0 \mid H_1]$$

Therefore for the minmax rule,

$$\begin{aligned} P_{e,H^*}(\pi_0) &= \pi_0 \mathbb{P}_{H^*}[\mathbf{y} \in \Gamma_1 \mid H_0] + (1-\pi_0)\mathbb{P}_{H^*}[\mathbf{y} \in \Gamma_0 \mid H_1] \\ &= \mathbb{P}_{H^*}[\mathbf{y} \in \Gamma_1 \mid H_0] = \mathbb{P}_{H^*}[\mathbf{y} \in \Gamma_0 \mid H_1] \end{aligned}$$

is *independent* of π_0.
Hence $P_{e,H|x=x_0} = P_{e,H|x=x_1}$, *i.e.,* the error probability conditioned on the message are the same. Note that this was the regularity condition we used in Theorem 2.2.1. Hence regardless of the choice of π_0, the error probability (average) is the same! If $\pi_0^* = \frac{1}{2}$ (*i.e., p_x* is uniform), then the maximum likelihood rule is the robust detection rule as stated in Theorem 2.2.1. Note that this is not so if $\pi_0^* \neq \frac{1}{2}$, then the MAP rule for π_0^* becomes the robust detection rule. Also note that the minmax rule makes the performance of *all* priors as bad as the worst prior.
Note: If $\pi_0^* = \frac{1}{2}$, or $P^*_{\mathbf{X}}(\mathbf{x})$ is uniform then minmax is the same as ML, and this occurs in several cases.

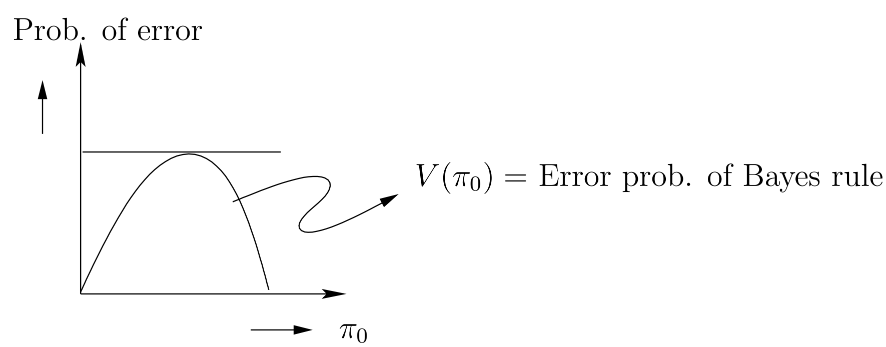


Figure 2.14: $P_{e,H}(\pi_0)$ Minmax detection rule.

Since minmax rule becomes Bayes rule for the *worst* prior, if the *worst* prior is uniform then clearly the minmax rule is the ML rule. Clearly if ML satisfies $\mathbb{P}^{ML}[error \mid H_j]$ independent of j then the ML rule is the robust detection rule.

2.2.3 Decision regions

Given the MAP and ML decision rules, we can divide $\mathbb{R}^N$ into regions which correspond to different decisions.
For example, in the AWGN case, the ML decoding rule will always decide to choose m_i if

$$\| \mathbf{y} - \mathbf{x}_i \|^2 < \| \mathbf{y} - \mathbf{x}_j \|^2, \ \forall j \neq i$$

Therefore, we can think of the region Γ_i as the decision region for m_i where,

$$\Gamma_i^{ML} = \{ \mathbf{y} \in \mathbb{R}^N \ : \| \mathbf{y} - \mathbf{x}_i \|^2 < \| \mathbf{y} - \mathbf{x}_j \|^2, \ \forall j \neq i \}$$

The MAP rule for the AWGN channel is a shifted region:

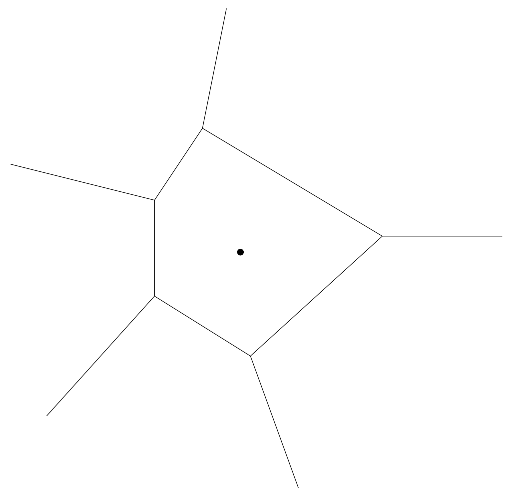


Figure 2.15: Voronoi regions for $\{x_i\}$, for uniform prior. Hence here the ML and MAP decision regions coincide.

$$\Gamma_i^{MAP} = \{ \mathbf{y} \in \mathbb{R}^N \ : \ \frac{\| \mathbf{y} - \mathbf{x}_i \|^2}{2\sigma^2} - \log[p_{\mathbf{X}}(\mathbf{x}_i)] < \frac{\| \mathbf{y} - \mathbf{x}_j \|^2}{2\sigma^2} - \log[p_{\mathbf{X}}(\mathbf{x}_j)], \ \forall j \neq i \}$$

The ML decision regions have a nice geometric interpretation. They are the *Voronoi* regions of the set of points $\{\mathbf{x}_i\}$. That is, the decision region associated with m_i is the set of all points in $\mathbb{R}^N$ which are closer to $\mathbf{x}_i$ than all the rest.
Moreover, since they are defined by Euclidean norms $\| \mathbf{y} - \mathbf{x}_i \|^2$, the regions are separated by *hyper planes*. To see this observe the decision regions are:

$$\begin{aligned}
\| \mathbf{y} - \mathbf{x}_i \|^2 &\leq \| \mathbf{y} - \mathbf{x}_j \|^2, \forall j \neq i \\
\Rightarrow -2 < \mathbf{y}, \mathbf{x}_i > + \| \mathbf{x}_i \|^2 &\leq -2 < \mathbf{y}, \mathbf{x}_j > + \| \mathbf{x}_j \|^2 \\
\Rightarrow < \mathbf{y}, \mathbf{x}_j - \mathbf{x}_i > &\leq \frac{1}{2}(\| \mathbf{x}_j \|^2 - \| \mathbf{x}_i \|^2) \\
\Rightarrow < \mathbf{y} - \frac{1}{2}(\mathbf{x}_j + \mathbf{x}_i), \mathbf{x}_i - \mathbf{x}_j > &\geq 0 \ \ \forall j \neq i
\end{aligned}$$

Hence the decision regions are bounded by hyperplanes since they are determined by a set of linear inequalities. The MAP decoding rule still produces decision regions that are hyper planes.

2.2.4 Bayes rule for minimizing risk

Error probability is just one possible criterion for choosing a detector. More generally the detectors minimize other cost functions. For example, let $\mathcal{C}_{i,j}$ denote the cost of choosing hypothesis i when actually hypothesis j was true. Then the expected **cost** incurred by some decision rule $H(\mathbf{y})$ is:

$$R_j(H) = \sum_i \mathcal{C}_{i,j} \mathbb{P}[H(\mathbf{Y}) = m_i \mid \mathcal{M} = m_j]$$

Therefore the overall average cost after taking prior probabilities into account is:

$$R(H) = \sum_j \mathbb{P}_{\mathbf{X}}(j) R_j(H)$$

Armed with this criterion we can ask the same question:
Question: *What is the optimal decision rule to minimize the above equation?*
Note: The error probability criterion corresponds to a cost assignment:

$$\mathcal{C}_{i,j} = 1,\ i \neq j,\ \mathcal{C}_{i,j} = 0,\ i = j.$$

Consider case M=2, *i.e.*, distinguishing between 2 hypotheses. Rewriting the equation for this case:

$$R(H) = \mathbb{P}_{\mathbf{x}}(0) R_0(H) + \mathbb{P}_{\mathbf{X}}(1) R_1(H)$$

where,

$$\begin{aligned} R_j(H) &= \mathcal{C}_{0,j} \mathbb{P}[H(\mathbf{Y}) = m_0 \mid \mathcal{M} = m_j] + \mathcal{C}_{1,j} \mathbb{P}[H(\mathbf{Y}) = m_1 \mid \mathcal{M} = m_j],\ j = 0, 1 \\ &= \mathcal{C}_{0,j} \{1 - \mathbb{P}[H(\mathbf{Y}) = m_1 \mid \mathcal{M} = m_j]\} + \mathcal{C}_{1,j} \mathbb{P}[H(\mathbf{Y}) = m_1 \mid \mathcal{M} = m_j],\ j = 0, 1 \end{aligned}$$

Let $P_{\mathbf{X}}(0) = \pi_0$, $P_{\mathbf{X}}(1) = 1 - \pi_0$

$$\begin{aligned} R(H) &= \pi_0 \mathcal{C}_{0,0} \mathbb{P}[\mathbf{y} \in \Gamma_0 \mid \mathbf{x} = x_0] + \pi_0 \mathcal{C}_{1,0} \mathbb{P}[\mathbf{y} \in \Gamma_1 \mid \mathbf{x} = x_0] \\ &+ \pi_1 \mathcal{C}_{0,1} \mathbb{P}[\mathbf{y} \in \Gamma_0 \mid \mathbf{x} = x_1] + \pi_1 \mathcal{C}_{1,1} \mathbb{P}[\mathbf{y} \in \Gamma_1 \mid \mathbf{x} = x_1] \\ &= \pi_0 \mathcal{C}_{0,0} - \pi_0 \mathcal{C}_{0,0} \mathbb{P}[\mathbf{y} \in \Gamma_1 \mid \mathbf{x} = x_0] + \pi_0 \mathcal{C}_{1,0} \mathbb{P}[\mathbf{y} \in \Gamma_1 \mid \mathbf{x} = x_0] \\ &+ \pi_1 \mathcal{C}_{0,1} - \pi_1 \mathcal{C}_{0,1} \mathbb{P}[\mathbf{y} \in \Gamma_1 \mid \mathbf{x} = x_1] + \pi_1 \mathcal{C}_{1,1} \mathbb{P}[\mathbf{y} \in \Gamma_1 \mid \mathbf{x} = x_1] \\ &= \pi_0 \mathcal{C}_{0,0} + \pi_1 \mathcal{C}_{0,1} + \pi_0 (\mathcal{C}_{1,0} - \mathcal{C}_{0,0}) \int_{\mathbf{y} \in \Gamma_1} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x} = x_0) d\mathbf{y} \\ &+ \pi_1 (\mathcal{C}_{1,1} - \mathcal{C}_{0,1}) \int_{\mathbf{y} \in \Gamma_1} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x} = x_1) d\mathbf{y} \\ &= \sum_{j=0}^{1} \pi_j \mathcal{C}_{0,j} + \int_{\mathbf{y} \in \Gamma_1} \left[\sum_{j=0}^{1} \pi_j (\mathcal{C}_{1,j} - \mathcal{C}_{0,j}) P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x} = x_j) \right] d\mathbf{y} \end{aligned}$$

Now, just like in the alternate proof for the MAP decoding rule, (see (2.8)) we want to minimize the last term. As seen in Figure 2.10 this is done by collecting the negative area in the function $\sum_{j=0}^{1} \pi_j (\mathcal{C}_{1,j} - \mathcal{C}_{0,j}) P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x} = x_j)$ as a function of $\mathbf{y}$. Therefore we get the decision rule,

$$\Gamma_1 = \{\mathbf{y} \in \mathbb{R}^N : \sum_{j=0}^{1} \mathbb{P}_{\mathbf{X}}(j) (\mathcal{C}_{1,j} - \mathcal{C}_{0,j}) \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}_j) < 0\}$$

Likelihood ratio: Surprisingly, in all the detection criteria we have seen the *likelihood ratio* defined as,

$$\frac{\mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}_0)}{\mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}_1)}$$

seems to appear as a part of the decision rule.
For example, if $\mathcal{C}_{1,1} < \mathcal{C}_{0,1}$, then we have,

$$\Gamma_1 = \{\mathbf{y} \in \mathbb{R}^N : \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}_1) > \tau \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}_0)\}$$

where $\tau = \frac{\mathbb{P}_{\mathbf{X}}(0)(\mathcal{C}_{1,0}-\mathcal{C}_{0,0})}{\mathbb{P}_{\mathbf{X}}(1)(\mathcal{C}_{0,1}-\mathcal{C}_{1,1})}$
For $\mathcal{C}_{0,0} = \mathcal{C}_{1,1} = 0$ and $\mathcal{C}_{0,1} = \mathcal{C}_{1,0} = 1$, we get the MAP rule, *i.e.,* $\tau = \frac{\mathbb{P}_{\mathbf{X}}(0)}{\mathbb{P}_{\mathbf{X}}(1)}$ which minimizes *average* error probability.

2.2.5 Irrelevance and reversibility

An output may contain parts that do not help to determine the message. These irrelevant components can be discarded without loss of performance. This is illustrated in the following example.

Example 2.2.3. *As shown Figure 2.16 if $\mathbf{z}_1$ and $\mathbf{z}_2$ are independent then clearly $\mathbf{y}_2$ is irrelevant.*

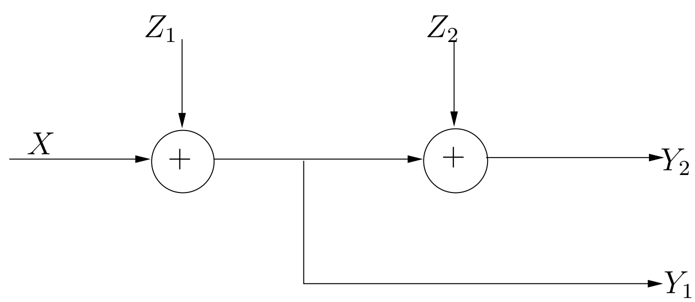


Figure 2.16: Example 2.2.3.

Theorem 2.2.2. *If $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$, and we have **either** of the following equivalent conditions:*

- $\mathbb{P}_{\mathbf{X}|\mathbf{Y}_1,\mathbf{Y}_2} = \mathbb{P}_{\mathbf{X}|\mathbf{Y}_1}$
- $\mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1,\mathbf{X}} = \mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1}$

*then $\mathbf{y}_2$ is **irrelevant** for the detection of $\mathbf{X}$.*

Proof: If $\mathbb{P}_{\mathbf{X}|\mathbf{Y}_1,\mathbf{Y}_2} = \mathbb{P}_{\mathbf{X}|\mathbf{Y}_1}$, then clearly the MAP decoding rule ignores $\mathbf{Y}_2$, and therefore it is irrelevant almost by definition. The question is whether the second statement is equivalent. Let $\mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1,\mathbf{X}} = \mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1}$

$$\mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1} = \frac{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2}}{\mathbb{P}_{\mathbf{Y}_1}} \tag{2.11}$$

$$\mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1,\mathbf{X}} = \frac{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2|\mathbf{X}}}{\mathbb{P}_{\mathbf{Y}_1|\mathbf{X}}} \tag{2.12}$$

Hence,

$$\begin{aligned}
\mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1,\mathbf{X}} = \mathbb{P}_{\mathbf{Y}_2|\mathbf{Y}_1} \Leftrightarrow \frac{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2}}{\mathbb{P}_{\mathbf{Y}_1}} &= \frac{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2|\mathbf{X}}}{\mathbb{P}_{\mathbf{Y}_1|\mathbf{X}}} \\
\Leftrightarrow \frac{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2|\mathbf{X}}}{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2}} &= \frac{\mathbb{P}_{\mathbf{Y}_1|\mathbf{X}}}{\mathbb{P}_{\mathbf{Y}_1}} \\
\Leftrightarrow \frac{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2|\mathbf{X}}\mathbb{P}_{\mathbf{X}}}{\mathbb{P}_{\mathbf{Y}_1,\mathbf{Y}_2}} &= \frac{\mathbb{P}_{\mathbf{Y}_1|\mathbf{X}}\mathbb{P}_{\mathbf{X}}}{\mathbb{P}_{\mathbf{Y}_1}} \\
\Leftrightarrow \mathbb{P}_{\mathbf{X}|\mathbf{Y}_1,\mathbf{Y}_2} &= \mathbb{P}_{\mathbf{X}|\mathbf{Y}_1}
\end{aligned} \tag{2.13}$$

□

Note: The irrelevance theorem is summarized by a Markov chain relationship

$$X \leftrightarrow Y_1 \leftrightarrow Y_2$$

which means that conditioned on Y_1, Y_2 is independent of X.

Application of Irrelevance theorem

Theorem 2.2.3. ***(Reversibility theorem)*** *The application of an invertible mapping on the channel output vector* $\mathbf{y}$*, does not affect the performance of the MAP detector.*

Proof: Let $\mathbf{y}_2$ be the channel output, and $\mathbf{y}_1 = G(\mathbf{y}_2)$, where $G(\cdot)$ is an invertible map. Then $\mathbf{y}_2 = G^{-1}(\mathbf{y}_1)$. Clearly

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ G^{-1}(\mathbf{y}_1) \end{bmatrix}$$

and therefore,

$$\mathbb{P}_{\mathbf{X}|\mathbf{Y}_1,\mathbf{Y}_2} = \mathbb{P}_{\mathbf{X}|\mathbf{Y}_1}$$

and hence by applying the irrelevance theorem, we can drop $\mathbf{y}_2$.

□

2.2.6 Complex Gaussian Noise

Let $\mathbf{z}$ be real Gaussian noise *i.e.,* $\mathbf{Z} = (z_1 \dots z_n)$, and

$$P_z(z) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{\frac{-||z||^2}{2\sigma^2}}$$

Let Complex Gaussian random variable be $Z^c = R + jI$. R, I are real and imaginary components, (R, I) is jointly Gaussian.

$$\mathcal{K} = \begin{bmatrix} \mathbb{E}[R^2] & \mathbb{E}[RI] \\ \mathbb{E}[IR] & \mathbb{E}[I^2] \end{bmatrix}$$

$$R_z{}^{(c)} = \mathbb{E}[Z^c Z^{c*}] = \mathbb{E}[|Z^c|^2] = \mathbb{E}[R]^2 + \mathbb{E}[I]^2$$

$$\mathbb{E}[Z^c Z^c] = \mathbb{E}[R^2] + j^2\mathbb{E}[I^2] + 2j\mathbb{E}[RI] = \mathbb{E}[R^2] - \mathbb{E}[I^2] + 2j\mathbb{E}[RI]$$

Circularly symetric Gaussian random variable:

$$\mathbb{E}[Z^{(C)}Z^{(C)}] = 0 \Leftrightarrow \begin{array}{ll} \mathbb{E}[R^2] & = \mathbb{E}[I^2] \\ \mathbb{E}[RI] & = 0 \end{array}$$

For complex Gaussian random vectors:

$$\mathbb{E}[Z_i^{(C)} Z_j^{(C)*}] = \mathbb{E}[R_i R_j] + \mathbb{E}[I_i I_j] - j\mathbb{E}[R_i I_j] - j\mathbb{E}[R_j I_i]$$

Circularly symmetric: $\mathbb{E}[Z_i^{(C)} Z_j^{(C)*}] = 0$ for all i, j.
Complex noise processes arise due to passband systems, we will learn more on them shortly.

2.2.7 Continuous additive white Gaussian noise channel

Let us go through the entire chain for a continuous (waveform) channel.
Channel: $\mathbf{y(t) = x(t) + z(t), t \in [0, T]}$
Additive White Gaussian Noise: Noise process $z(t)$ is Gaussian and "white" *i.e.*,

$$\mathbb{E}\left[z(t)z(t-\tau)\right] = \frac{N_0}{2}\delta(\tau)$$

Vector Channel Representation: Let the basis expansion and vector encoder be represented as,

$$x(t) = \sum_{n=0}^{N-1} x_n \phi_n(t).$$

Therefore, one can write,

$$y(t) = \sum_{n=0}^{N-1} x_n \phi_n(t) + z(t)$$

Let

$$y_n =< y(t), \phi_n(t) >, \;\; z_n =< z(t), \phi_n(t) >, \;\; n = 0, \ldots, N-1$$

Consider vector model,

$$\mathbf{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} = \mathbf{x} + \mathbf{z}$$

Note:

$$\hat{z}(t) \stackrel{def}{=} \sum_{n=0}^{N-1} z_n \phi_n(t) \neq z(t) \Longrightarrow \hat{y}(t) \stackrel{def}{=} \sum_{n=0}^{N-1} y_n \phi_n(t) \neq y(t)$$

Lemma 2.2.1. ***(Uncorrelated noise samples)*** *Given* ***any*** *orthonormal basis functions $\{\phi_n(t)\}$, and white Gaussian noise $z(t)$. The coefficients $\{z_n\} =< z, \phi_n >$ of the basis expansion are Gaussian and independent and identically distributed, with variance $\frac{N_0}{2}$,* i.e. $\mathbb{E}[z_n z_k] = \frac{N_0}{2}\delta_{n-k}$.

Therefore, if we extend the orthonormal basis $\{\phi_n(t)\}_{n=0}^{N-1}$ to span $\{z(t)\}$, the coefficients of the expansion $\{z_n\}_{n=0}^{N-1}$ would be *independent* of the rest of the coefficients.
Let us examine,

$$y(t) = \hat{y}(t) + \tilde{y}(t) = \underbrace{\sum_{n=0}^{N-1} (x_n + z_n)\phi_n(t)}_{\hat{y}(t)} + \underbrace{z(t) - \hat{z}(t)}_{\tilde{y}(t)}$$

Therefore, in vector expansion, $\tilde{\mathbf{y}}$ is the vector containing basis coefficients from $\phi_n(t), n = N, \ldots$. These coefficients can be shown to be *irrelevant* to the detection of $\mathbf{x}$, and can therefore be dropped. Hence for the detection process the following vector model is sufficient.

$$\mathbf{y} = \mathbf{x} + \mathbf{z}$$

Now we are back in "familiar" territory. We can write the MAP and ML decoding rules as before. Therefore the MAP decision rule is:

$$H_{\text{MAP}}(\mathbf{y}) = \arg\min_i \left[\frac{||\mathbf{y} - \mathbf{x}_i||^2}{2\sigma^2} - \log[\mathbb{P}_{\mathbf{X}}(\mathbf{x}_i)] \right]$$

And the ML decision rule is:

$$H_{\text{ML}}(\mathbf{y}) = \arg\min_i \left[\frac{||\mathbf{y} - \mathbf{x}_i||^2}{2\sigma^2} \right]$$

Let $p_x(x_i) = \frac{1}{M}$ i.e. uniform prior.
Here $ML \equiv MAP \equiv$ optimal detector.

$$\bar{\mathbb{P}}_e = \sum_{i=0}^{M-1} \mathbb{P}_{e|x=x_i} \mathbb{P}_x(x_i) = 1 - \sum_{i=0}^{M-1} \mathbb{P}_{c|x=x_i} \mathbb{P}_x(x_i)$$

$$\bar{\mathbb{P}}_{e,ML}^{\text{uniform prior}} = \frac{1}{M} \sum_{i=0}^{M-1} \mathbb{P}_{e,ML|x=x_i} = 1 - \frac{1}{M} \sum_{i=0}^{M-1} \mathbb{P}_{c|x=x_i}$$

The error probabilities depend on chosen signal constellation. More soon...

2.2.8 Binary constellation error probability

$$\mathbf{Y} = \mathbf{X}_i + \mathbf{Z},\ i = 0, 1,\ \mathbf{Z} \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_N)$$

Hence, conditional error probability is:

$$\begin{aligned}
\mathbb{P}_{e,ML|\mathbf{x}=\mathbf{x}_0} &= \mathbb{P}[||\mathbf{y} - \mathbf{x}_0|| \geq ||\mathbf{y} - \mathbf{x}_1||], \quad \text{since } \mathbf{y} = \mathbf{x}_0 + \mathbf{z}, && (2.14) \\
\Rightarrow \mathbb{P}_{e,ML|\mathbf{x}=\mathbf{x}_0} &= \mathbb{P}[||\mathbf{z}|| \geq ||(\mathbf{x}_0 - \mathbf{x}_1) + \mathbf{z}||] \\
&= \mathbb{P}[||\mathbf{z}||^2 \geq ||(\mathbf{x}_1 - \mathbf{x}_0) - \mathbf{z}||^2] \\
&= \mathbb{P}[||\mathbf{z}||^2 \geq ||\mathbf{x}_1 - \mathbf{x}_0||^2 + ||\mathbf{z}||^2 - 2 < (\mathbf{x}_1 - \mathbf{x}_0), \mathbf{z} >] && (2.15) \\
&= \mathbb{P}[\frac{< (\mathbf{x}_1 - \mathbf{x}_0), \mathbf{z} >}{||\mathbf{x}_1 - \mathbf{x}_0||} \geq \frac{||\mathbf{x}_1 - \mathbf{x}_0||}{2}]
\end{aligned}$$

But $< (\mathbf{x}_1 - \mathbf{x}_0), \mathbf{z} >$ is a Gaussian i.e. $U = \frac{(\mathbf{x}_1 - \mathbf{x}_0)^t}{||\mathbf{x}_1 - \mathbf{x}_0||}\mathbf{Z}$ is Gaussian, with $\mathbb{E}[U] = 0$, $\mathbb{E}[|U|^2] = \sigma^2$

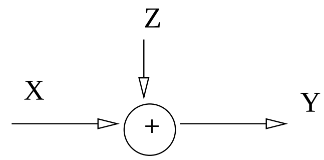
$$\begin{aligned}
\Rightarrow \mathbb{P}_{e,ML|\mathbf{x}=\mathbf{x}_0} &= \int_{\frac{||\mathbf{x}_1 - \mathbf{x}_0||}{2}}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-\frac{1}{2\sigma^2}||U||^2} dU && (2.16) \\
&= \int_{\frac{||\mathbf{x}_1 - \mathbf{x}_0||}{2\sigma}}^{\infty} \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{||\tilde{U}||^2}{2}} d\tilde{U} && (2.17) \\
&\stackrel{def}{=} Q\left(\frac{||\mathbf{x}_1 - \mathbf{x}_0||}{2\sigma}\right) && (2.18)
\end{aligned}$$

$$\Rightarrow \mathbb{P}_{e,ML} = \mathbb{P}_{\mathbf{X}}\{\mathbf{x}_0\}\mathbb{P}\{e, ML|\mathbf{x} = \mathbf{x}_0\} + \mathbb{P}_{\mathbf{X}}\{\mathbf{x}_1\}\mathbb{P}\{e, ML|\mathbf{x} = \mathbf{x}_1\} = Q\left(\frac{||\mathbf{x}_1 - \mathbf{x}_0||}{2\sigma}\right)$$

2.3 Error Probability for AWGN Channels

2.3.1 Discrete detection rules for AWGN

AWGN Channel: $Y = X + Z$, $Y \in \mathbb{C}^N$, $x \in \mathbb{C}^N$, $Z \sim \mathbb{C}$.



Let $p_x(x_i) = \frac{1}{M}$ i.e. uniform prior, hence the ML is equivalent to MAP detector.
Detection Rule:

$$\Gamma_i = \{y \in \mathbb{C}^N : ||y - x_i||^2 \leq ||y - x_j||^2, j \neq i\}$$

$$P_e = \sum_{i=0}^{M-1} P_{e|x=x_i} P_x(x_i) = 1 - \sum_{i=0}^{M-1} P_{c|x=x_i} P_x(x_i)$$

$$P_{e,ML}^{uniform\ prior} = \frac{1}{M} \sum_{i=0}^{M-1} P_{e,ML|x=x_i} = 1 - \frac{1}{M} \sum_{i=0}^{M-1} P_{c|x=x_i}$$

Hence, for $M > 2$, the error probability calculation could be difficult. We will develop properties and bounds that might help in this problem.

2.3.2 Rotational and translational invariance

Rotational Invariance

Theorem 2.3.1. *If all the data symbols are rotated by an orthogonal transformation, i.e.* $\widetilde{\mathbf{X}_i} = \mathbf{Q}\mathbf{x}_i$, $\forall i \in \{0, \ldots, M-1\}$*, where* $\mathbf{Q} \in \mathbb{C}^{N \times N}$, $\mathbf{Q}^*\mathbf{Q} = I$*, then the probability of error of the MAP/ML receiver remains unchanged over an AWGN channel.*

Proof: Let

$$\begin{aligned} \widetilde{\mathbf{Y}} &= \widetilde{\mathbf{X}} + \mathbf{Z} &(2.19)\\ \Rightarrow \underbrace{\mathbf{Q}^*\widetilde{\mathbf{Y}}}_{\mathbf{Y}} &= \underbrace{\mathbf{Q}^*\widetilde{\mathbf{X}}}_{\mathbf{X}} + \underbrace{\mathbf{Q}^*\mathbf{Z}}_{\widetilde{\mathbf{Z}}} &(2.20)\\ \Rightarrow \mathbf{Y} &= \mathbf{X} + \widetilde{\mathbf{Z}} \end{aligned}$$

but $\widetilde{Z}$ is Gaussian (linear transformation of Gaussian $\mathbf{Z}$) and $\mathbb{E}[\widetilde{\mathbf{Z}}\widetilde{\mathbf{Z}^*}] = \mathbf{Q}^*\sigma^2\mathbf{I}\mathbf{Q} = \sigma^2\mathbf{I} \Rightarrow \widetilde{\mathbf{Z}}$ is probabilistically equivalent to $\mathbf{Z} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.
Hence (2.19) is the same as $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ since $\mathbf{Q}$ is an invertible transform $\Rightarrow$ Probability of error is unchanged.

□

Translational Invariance

If all data symbols in a signal constellation are translated by constant vector amount, *i.e* $\widetilde{\mathbf{X}_i} = \mathbf{X}_i + \mathbf{a}, \forall i$ then the probability of error of the ML decoder remains the same on an AWGN channel.

Minimum energy translate: Substract $\mathbb{E}[\mathbf{X}]$ from every signal point. In other words, among equivalent signal constellations, a zero mean signal constellation has minimum energy.

2.3.3 Bounds for M > 2

As mentioned earlier, the error probability calculations for $M > 2$ can be difficult. Hence in this section we develop upper bounds for the error probability which is applicable for any constellation size M.

Theorem 2.3.2. ***Union bound***

$$\begin{aligned} P_{e,ML|x=x_i} &\leq \sum_{j\neq i} P_2(x_i, x_j) \\ &= \sum_{j\neq i} Q\left(\frac{||x_i - x_j||}{2\sigma}\right) \end{aligned}$$

$P_{e,ML} \leq (M-1)Q(\frac{d_{min}}{2\sigma})$ *where* $d_{min} \stackrel{def}{=} \min_{i\neq j} ||x_i - x_j||$

Proof: For $x = x_i$, i.e. $y = x_i + z$

$$\begin{aligned} P_{e,ML|x=x_i} &= \mathbb{P}\left[\bigcup_{j\neq i}\{||y - x_i|| > ||y - x_j||\}\right] \\ &\leq^{UB} \sum_{i\neq j} \mathbb{P}\left[||y - x_i|| > ||y - x_j||\right] = \sum_{i\neq j} P_2(x_i, x_j) \\ &\leq \sum_{i\neq j} Q\left(\frac{||x_i - x_j||}{2\sigma}\right) \\ &\leq (M-1)Q\left(\frac{d_{min}}{2\sigma}\right) \end{aligned}$$

since Q(.) is a monotonously decreasing function. Therefore

$$\begin{aligned} P_{e,ML} &= \sum_{i=0}^{M-1} P_x(x_i)P(e, ML|x = x_i) \\ &\leq \sum_{i=0}^{M-1} P_x(x_i)(M-1)Q\left(\frac{d_{min}}{2\sigma}\right) \\ &= (M-1)Q\left(\frac{d_{min}}{2\sigma}\right) \end{aligned}$$

□

Tighter Bound (Nearest Neighbor Union Bound)
Let N_i be the number of points sharing a decision boundary $\mathcal{D}_i$ with x_i.
Suppose x_k does not share a decision boundary with x_i, but $||y - x_i|| > ||y - x_k||$ then $\exists x_j \in \mathcal{D}_i$ s.t. $||y - x_i|| > ||y - x_j||$ where $\mathcal{D}_i$ is a set of points sharing the same decision boundary. Hence

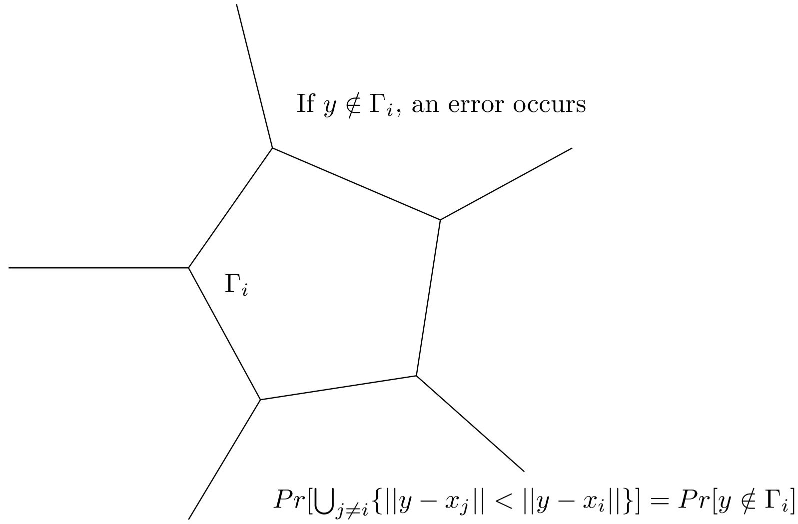


Figure 2.17: Decision regions for AWGN channel and error probability.

$||y - x_k|| < ||y - x_i|| \Rightarrow \exists x_j \in \mathcal{D}_i$ such that $||y - x_j|| < ||y - x_i||$

$$
\begin{aligned}
\Rightarrow \quad & \mathbb{P}[\bigcup_{j \neq i}\{||y - x_j|| < ||y - x_i||\}] \\
= \quad & \mathbb{P}[\bigcup_{j \in \mathcal{D}_i}\{||y - x_j|| < ||y - x_i||\}] \\
\leq \quad & N_i Q\left(\frac{d_{min}}{2\sigma}\right) \\
P_{e,ML} = \quad & \sum_{i=0}^{M-1} P_x(x_i) P(e, ML|x = x_i) \\
\leq \quad & \sum_{i=0}^{M-1} P_x(x_i) Q\left(\frac{d_{min}}{2\sigma}\right) N_i \\
\Rightarrow P_{e,ML} \leq \quad & N_e Q\left(\frac{d_{min}}{2\sigma}\right) \text{ where } N_e = \sum_i N_i P_x(x_i)
\end{aligned}
$$

Hence we have proved the following result,

Theorem 2.3.3. ***Nearest Neighbor Union bound (NNUB)***

$$P_{e,ML} \leq N_e Q(\frac{d_{min}}{2\sigma})$$

where

$$N_e = \sum N_i P_x(x_i)$$

and N_i is the number of constellation points sharing a decision boundary with x_i.

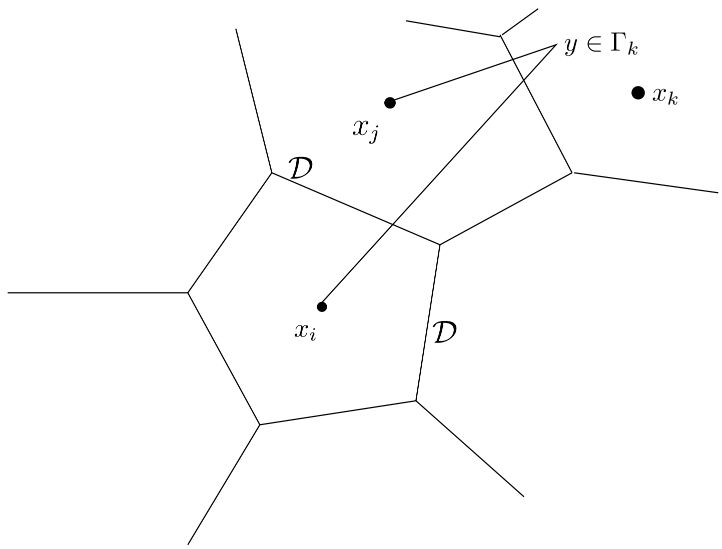


Figure 2.18: Figure illustrating geometry when $x_k \in \mathcal{D}_i$, then there is a $x_j \in \mathcal{D}_i$ such that **y** is closer to x_i.

2.4 Signal sets and measures

2.4.1 Basic terminology

In this section we discuss the terminology used *i.e.,* the rate, number of dimensions etc. and discuss what would be fair comparisons between constellations.
If signal bandwidth is *approximately* W and is *approximately* time-limited to T, then a deep theorem from signal analysis states that the space has dimension N which is

$$N = 2WT$$

If b bits are in a constellation in dimension N.

$$\begin{aligned}
\Rightarrow \bar{b} &= \frac{b}{N} = \#\text{ of bits/dimension} \\
R &= \text{rate} = \frac{b}{T} = \#\text{ bits/unit time} \\
\frac{R}{W} &= 2\bar{b} = \#\text{ bits/sec/Hz} \\
\bar{\mathcal{E}}_x &= \text{Average energy per dimension} = \frac{\mathcal{E}_x}{N} \\
P_x &= \text{Average power} = \frac{\mathcal{E}_x}{T}
\end{aligned}$$

$\bar{\mathcal{E}}_x$ useful in compound signal sets with different # of dimension.
Signal to noise ratio (SNR)

$$SNR = \frac{\mathcal{E}_x}{\sigma^2} = \frac{\text{Energy/dim}}{\text{Noise energy/dim}}$$

Constellation figure of merit (CFM)

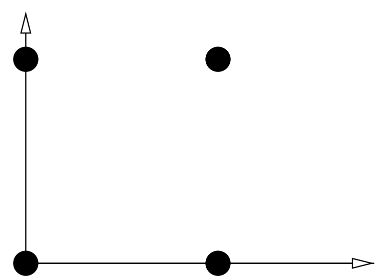
$$\zeta_x \stackrel{\text{def}}{=} \frac{(d_{min}/2)^2}{\bar{\mathcal{E}}_x}$$

As ζ_x increases we get better performance (for same # of bits per dimension *only*).
Fair comparison: In order to make a fair comparison between constellations, we need to make a multi-parameter comparison across the following measures.

Data rate (R)		bits/dim ($\bar{b}$)
Power (P_x)		Energy/dim ($\bar{\mathcal{E}}_x$)
Total BW (W)	OR	Normalized probability of error ($\bar{P}_e$)
Symbol period (T)		
Error probability (P_e)		

2.4.2 Signal constellations

Cubic constellations



$$\underline{x} = \sum_{i=0}^{N-1} U_i e_i$$

where N is the number of dimensions, and $e_i \in \mathbb{R}^N$ is,

$$e_i(k) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{else} \end{cases}$$

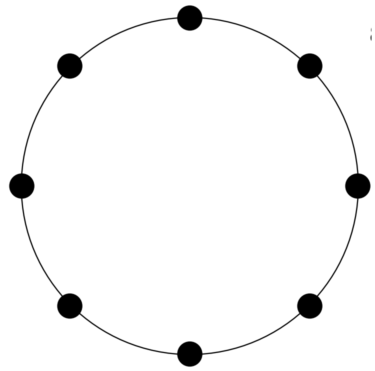
where $U_i \in \{0, 1\}$ depending on "bit sequence". Hence the number of constellation points is, $M = 2^N$.

Orthogonal constellations

$M = \alpha N$. Example: Bi-orthogonal signal set $\rightarrow M = 2N$ and $x_i = \pm e_i \Rightarrow 2N$ signal points.

Circular constellations

M^{th} root of unity



Example 2.4.1. ***Quadrature Phase-Shift Keying (QPSK):***

$$\begin{aligned}\phi_1(t) &= \sqrt{\frac{2}{T}}\cos(\frac{2\pi t}{T})\ \ 0 \le t \le T, \\ \phi_2(t) &= \sqrt{\frac{2}{T}}\sin(\frac{2\pi t}{T})\ \ 0 \le t \le T\end{aligned}$$

The constellation consists of $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, *where* $x_i \in \{-\sqrt{\frac{\mathcal{E}_x}{2}}, \sqrt{\frac{\mathcal{E}_x}{2}}\}$

$$\Rightarrow d_{min}^2 = 2\mathcal{E}_x,\ \zeta_x = \frac{[\frac{\sqrt{2\varepsilon_x}}{2}]^2}{\frac{\varepsilon_x}{2}} = 1.$$

Note that, $d_{min}^2 = 4\mathcal{E}_x$ *for BPSK.*

Error Probability:

$$\begin{aligned}\mathbb{P}_{correct} &= \sum_{i=0}^{3}\mathbb{P}_{correct|i}\mathbb{P}_x(i) = \mathbb{P}_{correct|0} \\ &= [1 - Q(\frac{d_{min}}{2\sigma})]^2 \\ \Rightarrow \mathbb{P}_{error} &= 2Q(\frac{d_{min}}{2\sigma}) - [Q(\frac{d_{min}}{2\sigma})]^2 < 2Q(\frac{d_{min}}{2\sigma}) \rightarrow NNUB\end{aligned}$$

Where $2Q(\frac{d_{min}}{2\sigma})$ *is the NNUB. Hence for* d_{min} *reasonably large the NNUB is tight.*

Example 2.4.2. ***M-ary Phase-Shift Keying (MPSK)***

$$d_{min} = 2\sqrt{\mathcal{E}_x}\sin(\frac{\pi}{M}),\ \zeta_x = \frac{[\sqrt{\mathcal{E}_x}\sin(\frac{\pi}{M})]^2}{\frac{\varepsilon_x}{2}} = 2\sin^2\frac{\pi}{M}$$

Error Probability: $\mathbb{P}_e < 2Q(\frac{\sqrt{\mathcal{E}_x}\sin(\frac{\pi}{M})}{\sigma})$

2.4.3 Lattice-based constellation:

A lattice is a "regular" arrangement of points in an N-dimensional space.

$$\mathbf{x} = \mathbf{Ga},\ a_i\ in\ \mathbb{Z}$$

where $\mathbf{G} \in \mathbb{R}^{N\times N}$ is called the generator matrix.

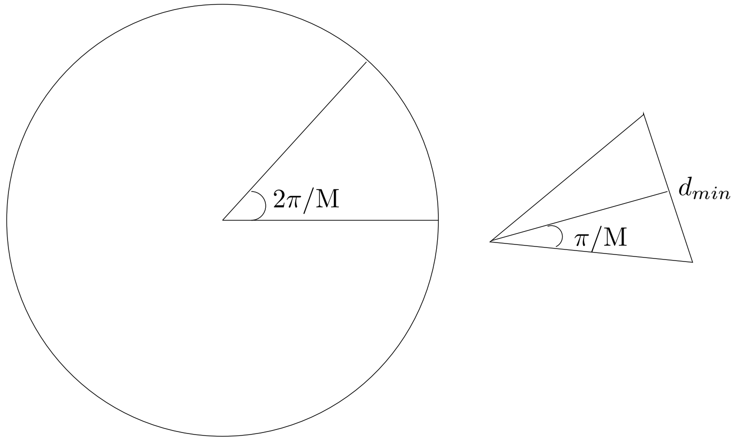


Figure 2.19: Figure for M-ary Phase-Shift keying.

Example 2.4.3. *Integer lattice:* $\mathbf{G} = \mathbf{I} \Rightarrow \mathbf{x} \in \mathbb{Z}^N$
If N=1 we get the "Pulse Amplitude Modulation" (PAM) constellation.
For this, $\mathcal{E}_x = \frac{d^2}{12}(M^2 - 1)$. *Thus,*

$$d_{min}^2 = \frac{12\mathcal{E}_x}{M^2 - 1}, \ \zeta_x = \frac{3\mathcal{E}_x}{M^2 - 1}$$

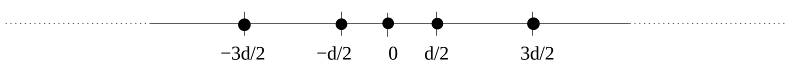


Figure 2.20: PAM constellation.

Error Probability:

$$\begin{aligned}
\mathbb{P}_{correct} &= \frac{M-2}{M}[1 - 2Q(\frac{d_{min}}{2\sigma})] + \frac{2}{M}[1 - Q(\frac{d_{min}}{2\sigma})] \\
\Rightarrow \mathbb{P}_e &= 2(1 - \frac{1}{M})Q(\frac{d_{min}}{2\sigma})
\end{aligned}$$

Number of nearest neighbors: $N_j = 2$ *for interior points, and* $N_j = 1$ *for end points.*

$$N_e = \frac{M-2}{M}2 + \frac{2}{M} = 2(1 - \frac{1}{M})$$

Note: *Hence NNUB is exact.*

Curious fact: For a given minimum distance d,

$$\begin{aligned}
M^2 &= 1 + \frac{12\mathcal{E}_x}{d^2} \\
\Rightarrow \bar{b} &= \log M = \frac{1}{2}\log(1 + \frac{12\mathcal{E}_x}{d^2})
\end{aligned}$$

Is this familiar? If so, is this a coincidence? More about this later...

Other lattice based constellations

Quadrature Amplitude Modulation (QAM): "Cookie-slice" of 2-dimensional integer lattice. Other constellations are carved out of other lattices (*e.g.* hexagonal lattice).

Other performance measures of interest

- **Coding gain:** $\gamma = \frac{\zeta_1}{\zeta_2}$
- Shaping gain of lattice.
- Peak-to-average ratio.

2.5 Problems

Problem 2.1

Consider a Gaussian hypothesis testing problem with $m = 2$. Under hypothesis $H = 0$ the transmitted point is equally likely to be $a_{00} = (1, 1)$ or $a_{01} = (-1, -1)$, whereas under hypothesis $H = 1$ the transmitted point is equally likely to be $a_{10} = (-1, 1)$ or $a_{11} = (1, -1)$. Under the assumption of uniform priors, write down the formula for the MAP decision rule and determine geometrically the decision regions.

Problem 2.2

[Minimax] Consider a scalar channel

$$Y = X + Z \tag{2.21}$$

where $X = \pm 1$ (i.e. $X \in \{-1, 1\}$) and $Z \sim \mathcal{N}(0, 1)$ (and Z is a **real** Gaussian random variable).

1. Let $\mathbb{P}[X = -1] = \frac{1}{2} = \mathbb{P}[X = 1]$, find the MAP decoding rule. Note that this is also the ML decoding rule. Now, let $\mathbb{P}[X = -1] = \Pi_0$ and $\mathbb{P}[X = 1] = 1 - \Pi_0$. Now, compute the error probability associated with the ML decoding rule as a function of Π_0. Given this calculation can you guess the worst prior for the MAP decoding rule? (*Hint:* You do **not** need to calculate $P_{e,MAP}(\Pi_0)$ for this)

2. Now, consider another receiver D_R, which implements the following decoding rule (for the same channel as in (2.21)).

   $$D_{R,1} = [\tfrac{1}{2}, \infty) \quad , \quad D_{R,-1} = (-\infty, \tfrac{1}{2})$$

   That is, the receiver decides that 1 was transmitted if it receives $Y \in [\frac{1}{2}, \infty)$ and decides that -1 was transmitted if $Y \in (-\infty, \frac{1}{2})$.

   Find $P_{e,R}(\Pi_0)$, the error probability of this receiver as a function of $\Pi_0 = \mathbb{P}[X = -1]$. Plot $P_{e,R}(\Pi_0)$ as a function of Π_0. Does it behave as you might have expected?

3. Find $\max_{\Pi_0} P_{e,R}(\Pi_0)$, i.e. what is the worst prior for this receiver?

4. Find out the value Π_0 for which the receiver D_R specified in parts (2) and (3) corresponds to the MAP decision rule. In other words, find for which value of Π_0, D_R is optimal in terms of error probability.

Problem 2.3

Consider the binary hypothesis testing problem with MAP decoding. Assume that priors are given by $(\pi_0, 1-\pi_0)$.

1. Let $V(\pi_0)$ be average probability of error. Write the expression for $V(\pi_0)$.
2. Show that $V(\pi_0)$ is a concave function of π_0 i.e.
$$V(\lambda\pi_0 + (1-\lambda)\pi_0') \geq \lambda V(\pi_0) + (1-\lambda)V(\pi_0'),$$
for priors $(\pi_0, 1-\pi_0)$ and $(\pi_0', 1-\pi_0')$.
3. What is the implication of concavity in terms of maximum of $V(\pi_0)$ for $\pi_0 \in [0,1]$?

Problem 2.4

Consider the Gaussian hypothesis testing case with non uniform priors. Prove that in this case, if y_1 and y_2 are elements of the decision region associated to hypothesis i then so is $\alpha y_1 + (1-\alpha)y_2$, where $\alpha \in [0,1]$.

Problem 2.5

Suppose Y is a random variable that under hypothesis H_j has density
$$p_j(y) = \frac{j+1}{2}e^{-(j+1)|y|}, y \in \mathbb{R}, j = 0, 1.$$
Assume that costs are given by
$$C_{ij} = \begin{cases} 0 & \text{if } \ i = j, \\ 1 & \text{if } \ i = 1 \ \text{ and } \ j = 0, \\ 3/4 & \text{if } \ i = 0 \ \text{ and } \ j = 1. \end{cases}$$

1. Find the MAP decision region assuming equal priors.
2. Recall that average risk function is given by:
$$R_H(\pi_0) = \sum_{j=0}^{1} \pi_j C_{0,j} + \sum_{j=0}^{1} \pi_j (C_{1,j} - C_{0,j}) P[H(\mathbb{Y}) = m_1 | M = m_j].$$
Assume that costs are given as above. Show that $R_{\text{MAP}}(\pi_0)$ is a concave function of π_0. Find the minimum, maximum value of $R_{\text{MAP}}(\pi_0)$ and the corresponding priors.

Problem 2.6

Consider the simple hypothesis testing problem for the real-valued observation Y:
$$H_0 : p_0(y) = \exp(-y^2/2)/\sqrt{2\pi}, \qquad y \in \mathbb{R}$$
$$H_1 : p_1(y) = \exp(-(y-1)^2/2)/\sqrt{2\pi}, \quad y \in \mathbb{R}$$
Suppose the cost assignment is given by $C_{00} = C_{11} = 0, C_{10} = 1$, and $C_{01} = N$. Find the minmax rule and risk. Investigate the behavior when N is very large.

Problem 2.7

Suppose we have a real observation Y and binary hypotheses described by the following pair of PDFs:

$$p_0(y) = \begin{cases} (1-|y|), & \text{if } |y| \leq 1 \\ 0, & \text{if } |y| > 1 \end{cases}$$

and

$$p_1(y) = \begin{cases} \frac{1}{4}(2-|y|), & \text{if } |y| \leq 2 \\ 0, & \text{if } |y| > 2 \end{cases}$$

Assume that the costs are given by

$$\begin{array}{rcccl} C_{01} & = & 2C_{10} & > & 0 \\ C_{00} & = & C_{11} & = & 0. \end{array}$$

Find the minimax test of H_0 versus H_1 and the corresponding minimax risk.

Problem 2.8

In the following a complex-valued random vector is defined as:

$$\mathbf{U} = \mathbf{U_R} + j\mathbf{U_I}$$

and we define the covariance matrix of a zero mean complex-valued random vector as :

$$K_{\mathbf{U}} = E[\mathbf{U}\mathbf{U}^\dagger]$$

We recall that a complex random vector is proper iff $K_{\mathbf{U}_R} = K_{\mathbf{U}_I}$ and $K_{\mathbf{U}_I\mathbf{U}_R} = -K^T_{\mathbf{U}_I\mathbf{U}_R}$. We want to prove that if $\mathbf{U}$ is a proper complex n-dimensional Gaussian zero mean random vector with covariance $\Lambda = E[\mathbf{U}\mathbf{U}^\dagger]]$, then the pdf of $\mathbf{U}$ is given by:

$$p_{\mathbf{U}}(\mathbf{u}) = \frac{1}{\pi^n \det(\Lambda)} \exp\{-\mathbf{u}^\dagger \Lambda^{-1} \mathbf{u}\}$$

1. Compute $\Phi = \mathrm{Cov}\left[\begin{bmatrix}\mathbf{U}_R \\ \mathbf{U}_I\end{bmatrix}, \begin{bmatrix}\mathbf{U}_R \\ \mathbf{U}_I\end{bmatrix}\right]$

2. A complex Gaussian random vector is defined as a vector with jointly Gaussian real and imaginary parts. Write $p_{\mathbf{U_R}\mathbf{U_I}}(\mathbf{u_R}, \mathbf{u_I})$.

3. Show the following lemma: Define the Hermitian $n \times n$ matrix $M = M_R + M_I + j(M_{IR} - M_{IR}^T)$ and the symmetric $2n \times 2n$ matrix $\Psi = 2\begin{bmatrix} M_R & M_{RI} \\ M_{IR} & M_I \end{bmatrix}$, then the quadratic forms $\mathcal{E} = \mathbf{u}^\dagger M \mathbf{u}$ and $\mathcal{E}' = \begin{bmatrix}\mathbf{u}_R^T & \mathbf{u}_I^T\end{bmatrix} \Psi \begin{bmatrix}\mathbf{u}_R \\ \mathbf{u}_I\end{bmatrix}$ are equal for all $\mathbf{u} = \mathbf{u}_R + j\mathbf{u}_I$ iff $M_I = M_R$ and $M_{IR} = -M_{IR}^T$

4. Suppose that $\Lambda^{-1} = \frac{1}{2}\Delta^{-1}(I - j\Lambda_{IR}\Lambda_R^{-1})$ where $\Delta = \Lambda_R + \Lambda_{IR}\Lambda_R^{-1}\Lambda_{IR}$. Apply the lemma given above to $\Psi = \frac{1}{2}\Phi^{-1}$ and $M = \frac{1}{2}\Delta^{-1}(I - j\Lambda_{IR}\Lambda_R^{-1})$ in order to show that $p_{\mathbf{U}}(\mathbf{u})$ and $p_{\mathbf{U_R}\mathbf{U_I}}(\mathbf{u_R}, \mathbf{u_I})$ have the same exponents. Use the matrix inversion formulae.

5. Show that $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BD^{-1}CD)$.

6. Using the result above show that $2^n\sqrt{\det\Phi} = \det\Lambda$

Problem 2.9

Consider the following signals:

$$\begin{aligned} x_0(t) &= \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{\pi}{6}\right) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \\ x_1(t) &= \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{5\pi}{6}\right) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \\ x_2(t) &= \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{3\pi}{2}\right) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(a) Find a set of orthonormal basis functions for this signal set. Show that they are orthonormal. *Hint*: Use the identity for $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

(b) Find the data symbols corresponding to the signals above using the basis functions you found in (a).

(c) Find the following inner products:

(i) $< x_0(t), x_0(t) >$

(ii) $< x_0(t), x_1(t) >$

(iii) $< x_0(t), x_2(t) >$

Problem 2.10

Consider an additive-noise channel $y = x + n$, where x takes on the values ± 3 with $P(x = 3) = 1/3$ and where n is a Cauchy random variable with PDF:

$$p_n(z) = \frac{1}{\pi(1+z^2)}.$$

Determine the decision regions of the MAP detector. Compare the decision regions found with those of the MAP detector for $n \sim \mathcal{N}(0,1)$. Compute the error probability in the two cases (Cauchy and Gaussian noise).

Problem 2.11

Consider the following constellation to be used on an AWGN channel with variance σ^2:

$$\begin{aligned} x_0 &= (-1,-1) \\ x_1 &= (1,-1) \\ x_2 &= (-1,1) \\ x_3 &= (1,1) \\ x_4 &= (0,3) \end{aligned}$$

1. Find the decision region for the ML detector.

2. Find the union bound and nearest neighbor union bound on P_e for the ML detector on this signal constellation.

Problem 2.12

A set of 4 orthogonal basis functions $\{\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)\}$ is used in the following constellation. In both the first 2 dimensions and again in the second two dimensions: The constellation points are restricted such that a point E can only follow a point E and a point O can only follow a point O. The points $\{1, 1\}$, $\{-1, -1\}$ are labeled as E and $\{1, -1\}$, $\{-1, 1\}$ are labeled as O points. For instance, the 4- dimensional point $[1, 1, -1, -1]$ is permitted to occur, but the point $[1, 1, -1, 1]$ can not occur.

1. Enumerate all M points as ordered-4-tuples.
2. Find b, $\bar{b}$.
3. Find $\mathcal{E}_x$ and $\bar{\mathcal{E}}_x$ (energy per dimension) for this constellation.
4. Find $d_{\min}$ for this constellation.
5. Find P_e and $\bar{P}_e$ for this constellation using the NNUB if used on an AWGN with $\sigma^2 = 0.1$.

Problem 2.13

Consider an additive-noise channel $y = x + n$, where x takes on the values $\pm A$ with equalprobability and where n is a Laplace random variable with PDF:

$$p_n(z) = \frac{1}{\sqrt{2}\sigma} e^{-|z|\sqrt{2}/\sigma}$$

Determine the decision regions of the MAP detector. Compare the decision regions found with those of the MAP detector for $n \sim \mathcal{N}(0, \sigma^2)$. Compute the error probability in the two cases (Laplace and Gaussian noise) and compare the resulting error probabilities for the same SNR (SNR is defined as $SNR = \frac{E[|x|^2]}{E[|n|^2]}$). What is the worst-case noise in the high SNR ?

Problem 2.14

Consider the general case of the 3-D Ternary Amplitude Modulation (TAM) constellation for which the data symbols are,

$$(x_l, x_m, x_n) = \left(\frac{d}{2}(2l - 1 - M^{\frac{1}{3}}), \frac{d}{2}(2m - 1 - M^{\frac{1}{3}}), \frac{d}{2}(2n - 1 - M^{\frac{1}{3}})\right)$$

with $l = 1, 2, \ldots, M^{\frac{1}{3}}$, $m = 1, 2, \ldots, M^{\frac{1}{3}}$, $n = 1, 2, \ldots, M^{\frac{1}{3}}$. Assume that $M^{\frac{1}{3}}$ is an even integer.

1. Show that the energy of this constellation is
$$\mathcal{E}_x = \frac{1}{M}\left[3M^{\frac{2}{3}} \sum_{l=1}^{M^{\frac{1}{3}}} x_l^2\right].$$
2. Now show that
$$\mathcal{E}_x = \frac{d^2}{4}(M^{\frac{2}{3}} - 1).$$
3. Find b and $\bar{b}$.
4. Find $\bar{\mathcal{E}}_x$ and the energy per bit $\mathcal{E}_b$.
5. For an equal number of bits per dimension $\bar{b} = \frac{b}{N}$, find the figure of merit for PAM, QAM and TAM constellations with appropriate sizes of M. Compare your results.

Problem 2.15

[BINARY COMMUNICATION CHANNEL]
Let $X \in \{0,1\}$ and $Z_i \in \{0,1\}$,

$$Y_i = X \oplus Z_i, i = 1, \ldots, n,$$

where $\oplus$ indicates a modulo-2 addition operation, i.e.,

$$0 \oplus 0 = 0, 0 \oplus 1 = 1, 1 \oplus 0 = 1, 1 \oplus 1 = 0.$$

Let X and Z_i be independent and for $\epsilon < 0.5$

$$X = \begin{cases} 0 & \text{w.p.} \quad \pi_0 \\ 1 & \text{w.p.} \quad \pi_1 \end{cases} \tag{2.22}$$

$$Z_i = \begin{cases} 0 & \text{w.p.} \quad 1-\epsilon \\ 1 & \text{w.p.} \quad \epsilon \end{cases} \tag{2.23}$$

$\{Z_i\}_{i=1}^n$ is an independent and identically distributed binary process, i.e.,

$$\mathbb{P}_{\underline{z}}(Z_1, \ldots, Z_n) = \prod_{i=1}^{n} \mathbb{P}_z(Z_i)$$

where $\mathbb{P}_z(Z_i)$ is specified in (2.23).

1. Given observations $\{Y_1, \ldots, Y_n\}$ find the MAP rule for detecting X. Note here that X is a scalar, i.e., a binary symbol transmitted n consecutive times over the channel. *Hint:* You can state the decision rule in terms of the number of ones in $\{Y_i\}$, i.e. in terms of $\sum_{i=1}^n Y_i$.

2. Find the error probability of detecting X as a function of the prior π_0.

3. What is the minmax rule for detecting X when the prior is unknown?

4. Assume now that $\pi_0 = \pi_1 = \frac{1}{2}$. Let

$$\boldsymbol{Y} = \underbrace{\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}}_{\boldsymbol{Y}} = \underbrace{\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}}_{\boldsymbol{X}} \oplus \underbrace{\begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}}_{\boldsymbol{Z}}$$

where $\oplus$ is again the modulo-2 addition aperator and the addition is done component-wise.

Here $\mathbf{X}$ is a vector and has the following two possible hypotheses

$$\boldsymbol{X} = \begin{cases} \mathbf{0} & \text{w.p. } \frac{1}{2} \quad \longrightarrow \quad \text{Hypothesis } H_0 \\ \boldsymbol{S} & \text{w.p. } \frac{1}{2} \quad \longrightarrow \quad \text{Hypothesis } H_1 \end{cases}$$

where $\mathbf{0} = [0, \ldots, 0]$ and $\boldsymbol{S} = [S_1, \ldots, S_n]$ is an i.i.d. process with

$$P_S(S_i) = \begin{cases} \frac{1}{2} & S_i = 0 \\ \frac{1}{2} & S_i = 1 \end{cases}.$$

Given n observations $Y_1, \ldots, Y_n$, we want to decide between the two hypotheses. Find the maximum likelihood rule to decide if $\mathbf{X} = \mathbf{0}$ or $\mathbf{X} = \mathbf{S}$, i.e. hypothesis H_0 or H_1. Again, you can state the decision rule in terms of the number of ones in $\{Y_i\}$.

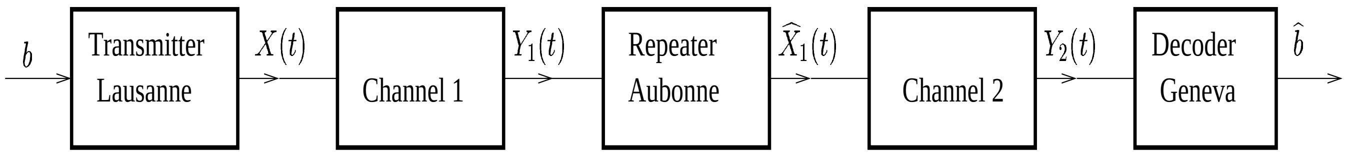


Figure 2.21: Block diagram for repeater channel

Problem 2.16

[Repeater channel]
We want to lay down a communication line from Lausanne to Geneva. Unfortunately due to the distance, we need a repeater in Aubonne. Luckily we get to design the repeater in Aubanne and use this to transmit the signal $\widehat{X}_1(t)$. In Geneva we are left with the task of detecting the transmitted bits. Let

$$\begin{aligned} Y_1(t) &= X(t) + Z_1(t), \\ Y_2(t) &= \widehat{X}_1(t) + Z_2(t). \end{aligned}$$

We assume that $Z_1(t)$ and $Z_2(t)$ are independent and identically distributed real zero-mean Gaussian noise processes with $\mathbb{E}\left[Z_1(t)^2\right] = \mathbb{E}\left[Z_2(t)^2\right] = \sigma^2$. Let

$$X(t) = b\phi(t).$$

where $\phi(t)$ is a normalized function, i.e., $\int |\phi(t)|^2 dt = 1$ and

$$b = \begin{cases} \sqrt{\mathcal{E}_x} & \text{w.p. } \frac{1}{2} \\ -\sqrt{\mathcal{E}_x} & \text{w.p. } \frac{1}{2} \end{cases}$$

1. Let the receiver is Aubonne attempt to detect the bit using a ML detector and then sends the signal

$$\widehat{X}_1(t) = \widehat{b}_1\phi(t), \tag{2.24}$$

   on wards to Geneva. Let it use a decoder such that

$$\begin{aligned} p_1 &= \mathbb{P}\left[\widehat{b}_1 = \sqrt{\mathcal{E}_x} | b = -\sqrt{\mathcal{E}_x}\right] \\ p_0 &= \mathbb{P}\left[\widehat{b}_1 = -\sqrt{\mathcal{E}_x} | b = \sqrt{\mathcal{E}_x}\right] \end{aligned}$$

   where $0 < p_0, p_1 < \frac{1}{2}$. Find the decoder in Geneva that minimizes

$$\mathbb{P}(\text{error}) = \mathbb{P}\left[\widehat{b} \neq b\right]$$

   and find an expression for the minimum $\mathbb{P}$(error) in terms of p_0 and p_1.

2. Show that to minimize $\mathbb{P}$(error) the decoder in Aubonne must be chosen to minimize $\mathbb{P}\left(\widehat{b}_1 \neq b\right)$. Specify the optimal decoder in Aubonne and the overall error probability in Geneva, i.e.,

$$\mathbb{P}\left(\widehat{b} \neq b\right),$$

   given this decoder in Aubonne.

Chapter 3

Passband Systems

In most communication systems the transmission occurs at a frequency band which is not at base band, but centered at a higher frequency. An example is that of wireless transmission, where the signal is centered around 1GHz or more. Other examples include TV broadcast, cordless phones, satellite communication, etc. In order to understand transmission over such channels we study representations of passband systems.

3.1 Equivalent representations

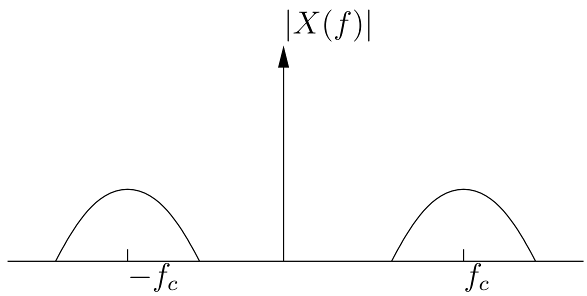


Figure 3.1: Passband transmission centered at frequency f_c.

Let carrier modulated signal $x(t)$ be given by,

$$x(t) = a(t)\cos[\omega_c t + \theta(t)]$$

the quadrature decomposition is

$$x(t) = \underbrace{x_I(t)\cos(\omega_c t)}_{in-phase} - \underbrace{x_Q(t)\sin(\omega_c t)}_{quadrature-phase}$$

Thus, $a(t) = \sqrt{x_I^2(t) + x_Q^2(t)}$, $\theta(t) = \tan^{-1}\left[\frac{x_Q(t)}{x_I(t)}\right]$.
The baseband-equivalent signal is

$$x_{bb}(t) \stackrel{def}{=} x_I(t) + jx_Q(t) \tag{3.1}$$

Note that in (3.1) there is no reference to ω_c.
The analytic equivalent signal is,

$$x_A(t) = x_{bb}(t)e^{j\omega_c t}$$

Hence,

$$x(t) = \mathrm{Re}[x_A(t)]$$

Let

$$\breve{x}(t) = \mathrm{Im}[x_A(t)]$$

Then

$$x_{bb}(t) \quad = \quad x_A(t)e^{-j\omega_c t} = [x(t) + j\breve{x}(t)]e^{-j\omega_c t}$$

Hence

$$x_{bb}(t) \quad = \quad \underbrace{[x(t)\cos(\omega_c t) + \breve{x}(t)\sin(\omega_c t)]}_{x_I(t)} + j\,\underbrace{[\breve{x}(t)\cos(\omega_c t) - x(t)\sin(\omega_c t)]}_{x_Q(t)}$$

Representation of passband signals Equivalent representations for $x(t) = a(t)\cos[\omega_x(t) + \theta(t)]$ are,

1. Magnitude and phase: $a(t), \theta(t)$
2. In-phase and quadrature phase: $x_I(t), x_Q(t)$
3. Complex baseband equivalent: $x_{bb}(t)$
4. Analytic signal: $x_A(t)$

3.2 Frequency analysis

We assume that the signal bandwidth is such that

$$\left.\begin{array}{l} X_Q(\omega) = 0 \\ X_I(\omega) = 0 \end{array}\right\} |\omega| > \omega_c$$

$$x_A(t) = x_{bb}(t)e^{j\omega_c t} = x(t) + j\breve{x}(t)$$

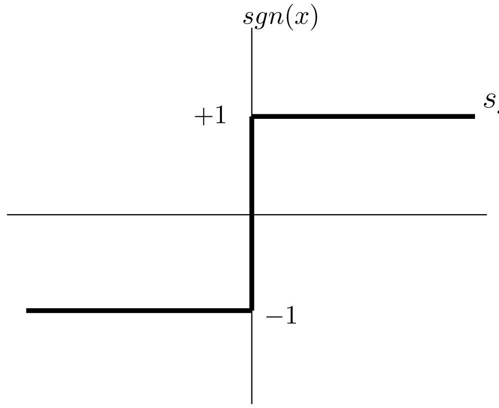
Now

$$\mathcal{F}[x(t)] = \mathcal{F}[x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t]$$

Where $\mathcal{F}(\cdot)$ is the Fourier transform.
Hence

$$X(\omega) = \frac{1}{2}[\ \underbrace{X_I(\omega - \omega_c)}_{\neq 0 \text{ only for} \omega > 0} \ + \ \underbrace{X_I(\omega + \omega_c)}_{\neq 0 \text{ only for } \omega < 0}\ ] - \frac{1}{2j}[\ \underbrace{X_Q(\omega - \omega_c)}_{\neq 0 \text{ only for } \omega > 0} \ - \ \underbrace{X_Q(\omega + \omega_c)}_{\neq 0 \text{ only for } \omega < 0}\ ]$$

Let sgn(x) be the function as defined in Figure 3.2.



$$sgn(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Figure 3.2: The sgn(·) function.

Now consider

$$\begin{aligned} -jsgn(\omega)X(\omega) &= \frac{-j}{2}[X_I(\omega-\omega_c) - X_I(\omega+\omega_c)] + \frac{j}{2j}[X_Q(\omega-\omega_c) + X_Q(\omega+\omega_c)] \\ &= \frac{1}{2j}[X_I(\omega-\omega_c) - X_I(\omega+\omega_c)] + \frac{1}{2}[X_Q(\omega-\omega_c) + X_Q(\omega+\omega_c)] \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{F}^{-1}\{-jsgn(\omega)X(\omega)\} &= x_I(t)\sin\omega_c t + x_Q(t)\cos\omega_c t = \mathrm{Im}[x_A(t)] \\ \Rightarrow \mathcal{F}\{\mathrm{Im}[x_A(t)]\} &= \mathcal{F}[\check{x}(t)] = \check{X}(\omega) = -jsgn(\omega)X(\omega) \end{aligned}$$

Hence we have

$$X_A(\omega) = X(\omega) + j(-jsgn(\omega))X(\omega) = [1 + sgn(\omega)]X(\omega)$$

Fourier relationships:

$$x(t) = x_I \cos\omega_c t - x_Q(t)\sin\omega_c t$$

$$\begin{aligned} x(t) &\overset{\mathcal{F}}{\Longleftrightarrow} X(\omega) \\ x_{bb}(t) = x_I(t) + jx_Q(t) &\overset{\mathcal{F}}{\Leftrightarrow} X_{bb}(\omega) \\ x_A(t) = x_{bb}(t)e^{j\omega_c t} &\overset{\mathcal{F}}{\Leftrightarrow} X_{bb}(\omega-\omega_c) = X_A(\omega) = [1 + sgn(\omega)]X(\omega) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad X_{bb}(\omega-\omega_c) &= [1 + sgn(\omega)]X(\omega) \\ \text{or} \quad X_{bb}(\omega) &= [1 + sgn(\omega+\omega_c)]X(\omega+\omega_c) = X_A(\omega+\omega_c) \end{aligned}$$

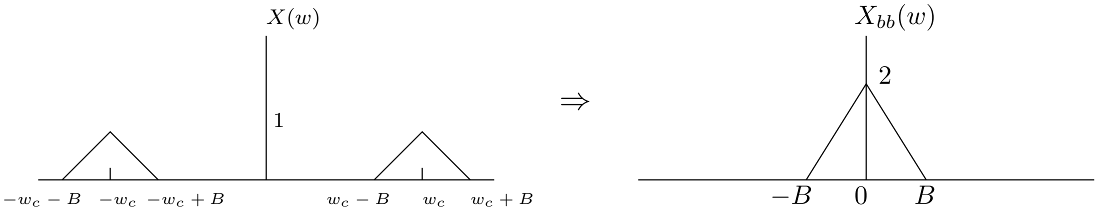


Figure 3.3: Passband and baseband signal representation.

3.3 Channel Input-Output Relationships

Given a passband channel $H(\omega)$, we can rewrite the channel output as,

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ \Rightarrow Y(\omega)[1 + sgn(\omega)] &= H(\omega)[1 + sgn(\omega)]X(\omega) \\ Y_A(\omega) &\overset{(a)}{=} H(\omega)X_A(\omega) \\ Y_A(\omega) &\overset{(b)}{=} \frac{1}{2}[1 + sgn(\omega)]H(\omega)X_A(\omega) \\ \Rightarrow Y_A(\omega) &= \frac{1}{2}H_A(\omega)X_A(\omega) \end{aligned}$$

Where (a) follows from derivation of $X_A(\omega)$, (b) is because $X_A(\omega)$ is non-zero only for $\omega > 0$ and hence $\frac{1}{2}[1 + sgn(\omega)] = 1$ for these frequencies. Now,

$$\begin{aligned} Y_{bb}(\omega) &= Y_A(\omega + \omega_c) = \frac{1}{2}H_A(\omega + \omega_c)X_A(\omega + \omega_c) \\ Y_{bb}(\omega) &= \frac{1}{2}H_{bb}(\omega)X_{bb}(\omega) \end{aligned}$$

Also,

$$Y_{bb}(\omega) = H(\omega + \omega_c)X_{bb}(\omega) \text{ for } \omega > -\omega_c$$

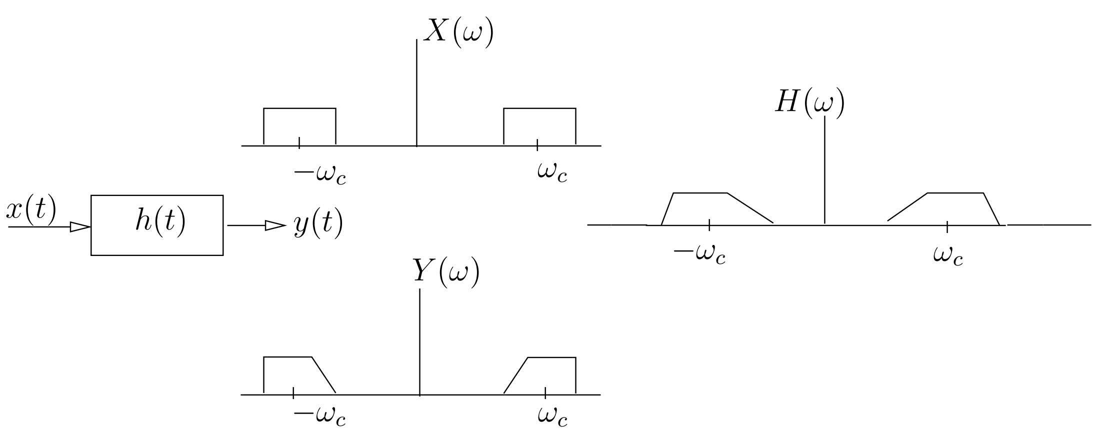


Figure 3.4: Representation for passband channels.

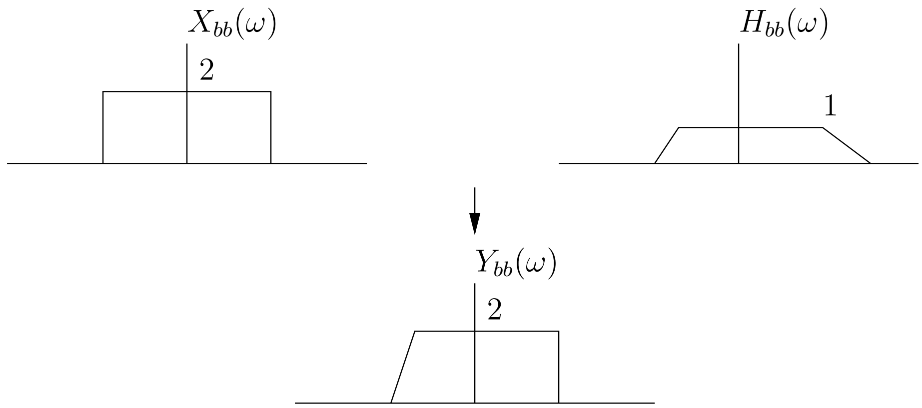


Figure 3.5: Representation with baseband signals.

SUMMARY of signal representations.

Passband	$y(t) = x(t) * h(t)$ $Y(\omega) = X(\omega)H(\omega)$
Analytic equivalent	$y_A(t) = x_A(t) * \frac{1}{2}h_A(t)$ $Y_A(\omega) = X_A(\omega)H(\omega) = X_A(\omega)\frac{1}{2}H_A(\omega)$
Baseband	$y_{bb}(t) = x_{bb}(t) * \frac{1}{2}h_{bb}(t)$ $Y_{bb}(\omega) = X_{bb}(\omega)H(\omega + \omega_c)$

3.4 Baseband equivalent Gaussian noise

Let the noise process be $z(t)$.

$$S_z(\omega) = \mathbb{E}[|Z(\omega)|]^2 = \begin{cases} \frac{N_0}{2} & \omega_c - W < |\omega| < \omega_c + W \\ 0 & \text{elsewhere} \end{cases}$$

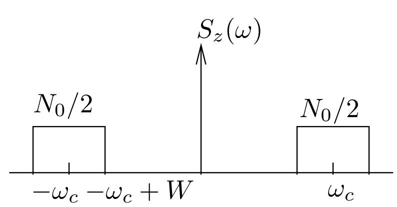


Figure 3.6: Bandlimited noise spectral density.

We can write,

$$z_A(t) = z(t) + j\check{z}(t).$$

The correlation function is,

$$\mathbb{E}[z(t)z(t-\tau)] = r_z(\tau)$$

Hence the power spectral density $S_z(\omega)$ is given by,

$$S_z(\omega) = \mathcal{F}\{r_z(\tau)\}$$

Note that formally $S_z(\omega) = \mathbb{E}[\mid Z(\omega)\mid^2]$, though these are technicalities which make this a formal rather than a precise relationship.
Therefore,

$$\begin{aligned} \mathbb{E}[|\breve{Z}(\omega)|^2] &= \mathbb{E}[|Z(\omega)|^2] = S_z(\omega) \\ S_{\breve{z}}(\omega) &= S_z(\omega) \end{aligned}$$

Hence we get the fact that

$$r_z(\tau) = r_{\breve{z}}(\tau) \tag{3.2}$$

almost everywhere (*i.e.,* except over a set of measure zero). Now let the Hilbert transform $\hbar(t)$ be,

$$\begin{aligned} \mathcal{H}(\omega) &= -jsgn(\omega) \\ \Updownarrow \mathcal{F}^{-1} & \\ \hbar(t) &= \begin{cases} \frac{1}{\pi t} & t \neq 0 \\ 0 & t = 0 \end{cases} \end{aligned}$$

This is because $\text{sgn}(\omega) = u(\omega) - u(-\omega)$ where $u(\cdot)$ is the unit-step function[1]. Since $\mathcal{F}^{-1}(u(\omega)) = -\frac{1}{j2\pi t} + \frac{1}{2}\delta(t)$, we can derive the expression for $\hbar(t)$. Hence we have,

$$\breve{z}(t) = \hbar(t) * z(t)$$

Now,

$$\begin{aligned} r_{z\breve{z}} &= \mathbb{E}[z(t)\breve{z}(t-\tau)] = \hbar^*(-\tau) * r_z(\tau) = -\hbar(\tau) * r_z(\tau) = -\breve{r}_z(\tau) \\ r_{\breve{z}z} &= \mathbb{E}[\breve{z}(t)z(t-\tau)] = \hbar(\tau) * r_z(\tau) = \breve{r}_z(\tau) \end{aligned}$$

Hence for,

$$z_A(t) = z(t) + j\breve{z}(t)$$

We have [2],

$$\begin{aligned} r_{z_A}(\tau) &= \mathbb{E}[z_A(t)z_A^*(t-\tau)] \\ &= \mathbb{E}[z(t)z^*(t-\tau)] - j\mathbb{E}[z(t)\breve{z}^*(t-\tau)] + \mathbb{E}[\breve{z}(t)z^*(t-\tau)] - j^2\mathbb{E}[\breve{z}(t)\breve{z}(t-\tau)] \\ &= r_z(\tau) - j(-\breve{r}_z(\tau)) + j\breve{r}_z(\tau) + r_z(\tau) \end{aligned}$$

Therefore we get,

$$r_{z_A}(\tau) = 2[r_z(\tau) + j\breve{r}_z(\tau)]$$

which implies

$$\begin{aligned} \mathcal{F}[r_{z_A}(\tau)] &= S_{z_A}(\omega) \\ &= 2[1 + sgn(\omega))]S_z(\omega) \\ &= \begin{cases} 4S_z(\omega) & \omega > 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

[1] The unit step function is $u(t) = 1, t > 0,\ u(t) = 0, t < 0$. At $t = 0$, there is a discontinuity, and usually $u(0) = \frac{1}{2}$ is used. Also $\mathcal{F}(u(t)) = \frac{1}{j\omega} + \pi\delta(\omega)$, where $\delta(\omega)$ is the Dirac delta function.

[2] We denote the complex conjugate by Z^*.

This gives us

$$S_{z_A}(\omega) = 4S_z(\omega) \text{ for } \omega > 0$$

that is,

$$S_{z_A}(\omega) = \begin{cases} 2N_0 & \omega_c - W \leq \omega \leq \omega_c + W \\ 0 & \text{elsewhere} \end{cases}$$

Since $z_A(t) = z_{bb}(t)e^{j\omega_c t}$, this implies that $r_{z_a}(\tau) = r_{z_{bb}}(\tau)e^{j\omega_c \tau}$, and we get (3.3),

$$\begin{aligned} S_{z_{bb}}(\omega) &= S_{z_A}(\omega + \omega_c) \\ \Rightarrow S_{z_{bb}}(\omega) &= \begin{cases} 2N_0 & |\omega| < W \\ 0 & \text{elsewhere} \end{cases} \end{aligned} \tag{3.3}$$

Here is where the messy factor of 2 arises!

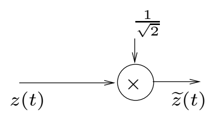


Figure 3.7: The factor of two in noise process.

Define,

$$\begin{aligned} \tilde{z}(t) &= \frac{1}{\sqrt{2}} z(t) \\ \Rightarrow S_{\tilde{z}}(\omega) &= \begin{cases} \frac{N_0}{4} & |\omega| < \omega_c + W \\ 0 & \text{elsewhere} \end{cases} \\ \Rightarrow S_{\tilde{z}_A}(\omega) &= \begin{cases} N_0 & \omega_c - W \leq W \leq \omega_c + W \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

and

$$\Rightarrow S_{\tilde{z}_{bb}}(\omega) = \begin{cases} N_0 & |\omega| < W \\ 0 & \text{elsewhere} \end{cases}$$

giving the same energy as the passband noise spectrum.

Example 3.4.1. *Let*

$$x_{bb}(t) = \sqrt{2}(x_1 + jx_2)\varphi(t)$$

if

$$x(t) \quad = x_1\varphi_1(t) + x_2\varphi_2(t)$$

where,

$$\begin{aligned} \varphi_1(t) &= \sqrt{2}\varphi(t)\cos\omega_c t \\ \varphi_2(t) &= -\sqrt{2}\varphi(t)\sin\omega_c t \end{aligned}$$

If $\varphi(t)$ is normalized, i.e. $||\varphi|| = 1$*, then if*

$$\varphi_1(t) = \sqrt{2}\varphi(t)\cos\omega_c t,\ \varphi_2(t) = \sqrt{2}\varphi(t)\sin\omega_c t$$

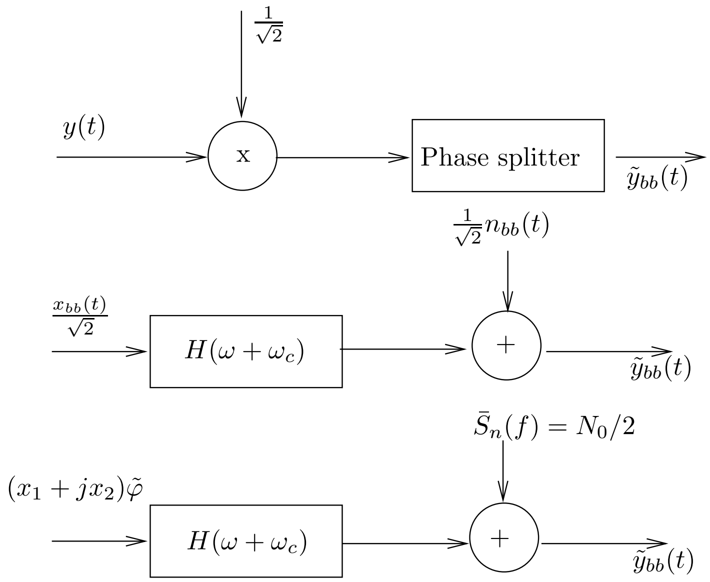


Figure 3.8: Representation of additive noise channel.

$$\Rightarrow ||\varphi_1|| = ||\varphi_2|| = 1$$

So under modulation $\sqrt{2}$ factor is needed.
Verification: *Let us verify that indeed φ_1 and φ_2 are normalized.*

$$\Phi_1(\omega) = \sqrt{2}[\frac{1}{2}\Phi(\omega-\omega_c) + \frac{1}{2}\Phi(\omega+\omega_c)] = \frac{1}{\sqrt{2}}[\Phi(\omega-\omega_c) + \Phi(\omega+\omega_c)],$$

where $\Phi_i(\omega) = \mathcal{F}(\varphi_i), i = 1, 2$. Hence, it is normalized, i.e.,

$$||\Phi_1|| = \frac{1}{2}[||\Phi|| + ||\Phi||] = 1$$

Note: Scaling is really for analytical convenience, since scaling does *not* change SNR since it is after received signal.
Therefore to be precise when $X_A(w) = [1 + sgn(w)]X(w)$

$$\Rightarrow X_A(\omega) = \begin{cases} 2X(\omega) & \omega > 0 \\ X(\omega) & \omega = 0 \\ 0 & \omega < 0 \end{cases}$$

3.5 Circularly symmetric complex Gaussian processes

Let $Z = R + jI$ where I and R denote real and imaginary components respectively.

$$Z \text{ complex Gaussian } \Rightarrow R, I \text{ are jointly Gaussian}$$

Now if we think of $\begin{pmatrix} R \\ I \end{pmatrix}$ as a vector then

$$\mathbb{E}\left[\begin{pmatrix} R \\ I \end{pmatrix}\begin{pmatrix} R & I \end{pmatrix}\right] = \begin{pmatrix} \mathbb{E}[R^2] & \mathbb{E}[RI] \\ \mathbb{E}[IR] & \mathbb{E}[I^2] \end{pmatrix}$$

Note that since $\mathbb{E}[IR] = \mathbb{E}[RI]$, there are three degrees of freedom.
Let

$$\mathcal{R}_z = \mathbb{E}[ZZ^*] = \mathbb{E}\left[R^2\right] + \mathbb{E}\left[I^2\right]$$

Clearly this has only *one* degree of freedom. Hence, this is *not* sufficient to specify the Gaussian random variable Z.
Define

$$\widetilde{\mathcal{R}}_z = \mathbb{E}[ZZ] = \mathbb{E}\left[R^2\right] - \mathbb{E}\left[I^2\right] + 2j\mathbb{E}[RI]$$

Thus we have 2 degrees of freedom here.

Definition 3.5.1. ***Circularly symmetric complex Gaussian variables.*** *A random variable Z is circularly symmetric iff* $\Rightarrow \widetilde{\mathcal{R}}_z = 0$, *i.e.* $\begin{cases} \mathbb{E}\left[R^2\right] = \mathbb{E}\left[I^2\right] \\ \mathbb{E}[RI] = 0 \end{cases}$

This can be generalized to vectors,

$$\mathbf{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \in \mathbb{C}^N$$

is a complex Gaussian circularly symmetric random variable, iff,

$$\mathbb{E}\left[ZZ^t\right] \quad = \quad \mathbf{0}$$

This implies,

$$\mathbb{E}[z_i z_j] \quad = \quad 0 \,\forall i, j$$

Therefore the complex covariance matrix R_z for zero mean random variables is,

$$\mathcal{R}_z = \mathbb{E}[\mathbf{z}\mathbf{z}^*]$$

Complex Gaussian vectors The probability density function (pdf) is,

$$P_z(z) = \frac{1}{|\pi \mathcal{R}_z|} e^{-(z-\mu)^* \mathcal{R}_z{}^{-1}(z-\mu)}$$

where $\mathbb{E}[z] = \mu$, and $\mathcal{R}_z = \mathbb{E}\left[(z-\mu)(z-\mu)^*\right]$.

3.5.1 Gaussian hypothesis testing - complex case

$$\mathbf{y} = \mathbf{x} + \mathbf{z}$$

where $\mathbf{x}$,$\mathbf{y}$,$\mathbf{z} \in \mathbb{C}^N$ and $\mathbf{z}$ is a circularly symmetric complex Gaussian random vector with $\mathcal{R}_z = \sigma^2 I$,

$\mu = 0$.

$$\begin{aligned}
H_{MAP}(y) &= \arg\max_i \mathbb{P}_{x|y}(\mathbf{x}_i|\mathbf{y}) \\
&= \arg\max_i \mathbb{P}_x(x_i)\mathbb{P}_{y|x}(\mathbf{y}|x_i) \\
&= \arg\max_i \mathbb{P}_x(x_i)\mathbb{P}_z(y-x_i) \\
&= \arg\max_i \mathbb{P}_x(x_i)\frac{1}{(\pi\sigma^2)^N}e^{-\frac{||y-x_i||^2}{\sigma^2}} \\
&= \arg\max_i \left[\ln \mathbb{P}_x(x_i) - \frac{||y-x_i||^2}{\sigma^2}\right] \\
&= \arg\max_i \left[\ln \mathbb{P}_x(x_i) - \frac{||y||^2}{\sigma^2} - \frac{||x_i||^2}{\sigma^2} + \frac{2\Re < y, x_i >}{\sigma^2}\right] \\
\Rightarrow H_{MAP}(y) &= \arg\max_i \left[\ln \mathbb{P}_x(x_i) + \frac{2\Re < y, x_i >}{\sigma^2} - \frac{||x_i||^2}{\sigma^2}\right]
\end{aligned}$$

where $< y, x >= y^* x$
Similarly

$$\begin{aligned}
H_{ML}(y) &= \arg\max_i [-||y-x_i||^2] \\
&= \arg\max_i [2\Re < y, x_i > -||x_i||^2] \\
&= \arg\min_i [||y-x_i||^2]
\end{aligned}$$

3.6 Problems

Problem 3.1

Let the transmitted bandpass signal be given by

$$x(t) = a\cos\left(2\pi(f_c + \tfrac{1}{T})t\right) + b\cos\left(2\pi(f_c + \tfrac{2}{T})t\right) \qquad t \in [0, T]$$

and $a \in \{0, A\}$, $b \in \{0, A\}$.

1. Find the baseband equivalent signal $x_{bb}(t)$ for the transmitted signal.
2. Find the vector representation of the baseband signal and draw the corresponding signal constellation.
3. If $a = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ A & \text{w.p. } \frac{1}{2} \end{cases}$ and $b = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ A & \text{w.p. } \frac{1}{2} \end{cases}$
find the average energy of the baseband signal. Is this a minimum energy configuration? If not how will you modify the constellation so that it is of minimum energy?

Problem 3.2

Consider the following passband waveform:

$$x(t) = sinc(t)(1 + A\sin(4\pi t))\cos\left(\omega_c t + \frac{\pi}{4}\right),$$

where $\omega_c >> 4\pi$.

1. Find $x_{bb}(t)$ and $x_A(t)$, the baseband and analytical equivalents of $x(t)$.
2. Let $x(t)$ is passed through a channel with impulse response $h(t) = \delta(t) - \delta(t-1)$. Let the output be $y(t)$. Find $y_{bb}(t)$ and $y_A(t)$.

Problem 3.3

[PASSBAND SYSTEMS] Let the transmitted bandpass signal be given by

$$x(t) = a\cos\left(2\pi(f_c + \tfrac{1}{T})t\right) + b\cos\left(2\pi(f_c + \tfrac{2}{T})t\right) \qquad t \in [0, T]$$

and $a \in \{0, A\}$, $b \in \{0, A\}$.

1. Find the baseband equivalent signal $x_{bb}(t)$ for the transmitted signal.
2. Find the vector representation of the baseband signal and draw the corresponding signal constellation.
3. If $a = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ A & \text{w.p. } \frac{1}{2} \end{cases}$ and $b = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ A & \text{w.p. } \frac{1}{2} \end{cases}$
   find the average energy of the baseband signal. Is this a minimum energy configuration? If not how will you modify the constellation so that is is minimum energy?

Problem 3.4

A baseband-equivalent waveform $(w_c > 2\pi)$

$$\tilde{x}_{bb}(t) = (x_1 + jx_2)\text{sinc}(t)$$

is convolved with the complex filter

$$w_1(t) = \delta(t) - j\delta(t-1)$$

1. Find
   $$y(t) = w_1(t) * \tilde{x}_{bb}(t).$$
2. Suppose $y(t)$ is convolved with the imaginary filter
   $$w_2(t) = 2j\text{sinc}(t)$$
   to get
   $$\begin{aligned} z(t) &= w_2(t) * y(t) \\ &= w_2(t) * w_1(t) * \tilde{x}_{bb}(t) \\ &= w(t) * \tilde{x}_{bb}(t). \end{aligned}$$
   Find $z(t)$. Note that $\text{sinc}(t) * \text{sinc}(t-k) = \text{sinc}(t-k)$, k an integer.
3. Let
   $$\tilde{z}(t) = \Re\{z(t)e^{jw_ct}\} = \tilde{w}(t) * x(t)$$
   where $x(t) = \Re\{\tilde{x}_{bb}(t)e^{jw_ct}\}$. Show that
   $$\tilde{w}(t) = 4\,\text{sinc}(t-1)\cos(w_ct) - 4\,\text{sinc}(t)\sin(w_ct)$$
   when convolved with the passband $x(t)$ will produce $\tilde{z}(t)$. Hint: use baseband calculations.

Problem 3.5

For the AWGN channel with transfer function,

$$H(f) = \begin{cases} 1 & 75\text{MHz} \leq |f| \leq 175\text{MHz} \\ 0 & \text{otherwise} \end{cases}$$

a transmitted signal cannot exceed 1 mW and the power spectral density is also limited according to $S_x(f) \leq -83$ dBm/Hz (two sided psd). The two-sided noise power spectral density is $\sigma^2 = -98$ dBm/Hz. The carrier frequency is $f_c = 100$ MHz for QAM transmission. The probability of error is $P_e = 10^{-6}$. (P dBm $= 10^{P/10}$ mW.)

1. Find the base band channel model $\frac{H_{bb}(f)}{2}$ (after the scaling).
2. Find the largest symbol rate that can be used with the 100 MHz carrier frequency.
3. What is the maximum signal power at the channel output with QAM?
4. What QAM data rate can be achieved with the symbol rate of part 2?

Problem 3.6

[WHITENING PASSBAND RANDOM PROCESSES]
Let us consider a real valued passband random process

$$X(t) = X_I(t)\cos(\omega_c t) - X_Q(t)\sin(\omega_c t), \tag{3.4}$$

at a carrier frequency of ω_c. The processes $\{X_I(t)\}$ and $\{X_Q(t)\}$ are independent stationary zero-mean Gaussian random processes with,

$$\begin{aligned} \mathbb{E}\left[X_I(t)X_I(t-\tau)\right] &= r_I(\tau) \\ \text{and} \quad \mathbb{E}\left[X_Q(t)X_Q(t-\tau)\right] &= r_Q(\tau). \end{aligned}$$

We assume that

$$\begin{aligned} r_I(\tau) = r_Q(\tau) &= e^{-2|\tau|} \\ \text{and} \quad \mathbb{E}\left[X_I(t)X_Q(t-\tau)\right] &= 0 \quad \forall\tau. \end{aligned}$$

1. Find $r_x(\tau) = \mathbb{E}\left[X(t)X(t-\tau)\right]$ in terms of $r_I(\tau) = r_Q(\tau)$ as a function of ω_c.
2. Find the correlation function for the baseband equivalent of $X(t)$ in equation (3.4), i.e., find

   $$\mathbb{E}\left[X_{bb}(t)X_{bb}^*(t-\tau)\right] = r_{bb}(\tau).$$

   Also find the correlation function of the analytic equivalent signal $\mathbb{E}\left[X_A(t)X_A^*(t-\tau)\right]$.
3. We sample $X_{bb}(t)$ at times $t = k$, i.e., at integer times,

   $$Y(k) = X_{bb}(k).$$

   Hence

   $$\mathbb{E}\left[Y(k)Y^*(k-l)\right] = \mathbb{E}\left[X_{bb}(k)X_{bb}^*(k-l)\right] = r_{bb}(l).$$

   We want to find a baseband filter to whiten this $\{Y(k)\}$ process. Compute such a causal whitening filter.

Part II

Transmission over Linear Time-Invariant channels

Chapter 4

Inter-symbol Interference and optimal detection

The main focus till now was one "one-shot" transmission, *i.e.,* transmission of one of M messages using a N dimensional symbol. Typically, a transmission system sends a sequence of messages. If the channel is *memoryless*, *i.e.,* successive transmissions do not interfere with each other, then the analysis up to now is sufficient. More formally, a channel is memoryless if

$$\mathbb{P}(y_1, y_2, \ldots, y_t | x_1, x_2, \ldots, x_t) = \prod_{k=1}^{t} \mathbb{P}(y_k | x_k)$$

where $\{y_k\}_{k=1}^{t}$ are symbols received over t successive time periods. In practice, symbols do interfere with each other causing Inter-Symbol-Interference (ISI).

4.1 Successive transmission over an AWGN channel

First, let us examine successive transmission over a memoryless AWGN channel. If we reuse the channel we send one out of M messages every T seconds. If T is the symbol period then $\frac{1}{T}$ is the symbol rate. Then the data rate R is defined as

$$R \stackrel{def}{=} \frac{\log_2 M}{T} bits/second.$$

In order to transmit k successive messages, we send

$$x(t) = \sum_{k=0}^{K-1} x_k(t - kT) \tag{4.1}$$

where k indicates time index (*i.e.,* message transmitted at kth time instant). We also define

$$x_k(t) = \sum_{n=0}^{N-1} x_{k,n} \phi_n(t)$$

Thus if we want to express $x(t)$ in terms of its basis functions we have,

$$x(t) = \sum_{k=0}^{K-1} x_k(t - kT) = \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} x_{k,n} \underbrace{\phi_n(t - KT)}_{\psi_{k,n}(t)} \tag{4.2}$$

Where the equivalent basis $\{\psi_{k,n}(t)\}$ has two indices, one indicating the symbol time and other for the basis coefficient, for the symbol time. One can form the optimal receiver by thinking of (4.1) as a one-shot problem with M^k possible messages and therefore decode optimally. If the channel is memoryless, the problem decouples and instead of M^k, we get k, M-dimensional independently solvable problems and the dimensionality of the problem does not go up exponentially *i.e.,* the complexity does not grow as M^k.
Notes:

1. Even though $< \varphi_n, \varphi_k >= \delta_{n-k}$, $< \varphi_n(t-kT), \varphi_k(t-lT) >$ need not be δ_{n-k} for $k \neq l$.

   However, if $< \varphi_n(t-kT), \varphi_k(t-lT) >= \delta_{n-k}$ again the problem decouples and we can do "symbol-by-symbol" detection.

2. The inherent assumption is that the successive messages are independent. However, if there is dependance, then clearly "symbol-by-symbol" detection is *not* optimal even for memoryless channels.

4.2 Inter-symbol Interference channel

The inter-symbol interference channel we will focus on is illustrated in Figure 4.1.

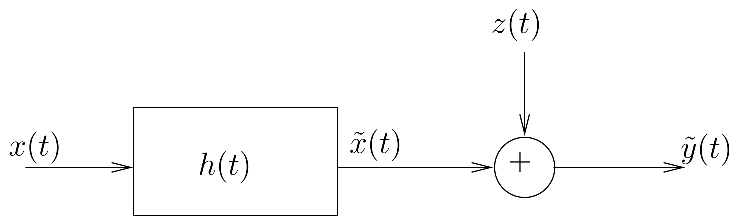


Figure 4.1: The inter symbol interference channel.

The transmitted waveform is,

$$\tilde{x}(t) = \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} x_{k,n} \underbrace{[\varphi_n(t-kT) * h(t)]}_{p_n(t-kT)},$$

and the received waveform is with Gaussian noise,

$$\tilde{y}(t) = \tilde{x}(t) + z(t) = \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} x_{k,n} \underbrace{[\varphi_n(t-kT) * h(t)]}_{p_n(t-kT)} + z(t), \tag{4.3}$$

where $z(t)$ is additive Gaussian noise.
Therefore the n^{th} pulse response is,

$$p_n(t) = \varphi_n(t) * h(t)$$

Definition 4.2.1. *We define the normalized pulse response* $\widetilde{\varphi}_n(t)$ *as,*

$$\widetilde{\varphi}_n(t) = \frac{p_n(t)}{||p_n||}$$

where $||p_n|| = < p_n, p_n >^{1/2}$.

Hence, we can rewrite the transmitted waveform as,

$$\tilde{x}(t) = \sum_{k=0}^{K-1}\sum_{n=0}^{N-1} \underbrace{x_{k,n}||p||}_{\text{scaled version of symbol}} \widetilde{\varphi}_n(t-kT)$$

Now consider an isolated symbol transmitted over the channel, *i.e.,* not successive transmission. This means that $\mathbf{x}_k = \mathbf{x}_0\delta(k)$, and the waveform $y(t)$ is observed in order to detect $\mathbf{x}_0$. Therefore, this "one-shot" transmission specifies a lower bound to the error probability of i.i.d. symbols. Therefore the SNR of this channel is an upper bound to the SNR of successive data transmission over the ISI channel. Therefore, this is called the **matched filter bound**.

Definition 4.2.2. *The SNR_{MFB} is the SNR of a single shot transmission over the channel given in (4.3) and is given by,*

$$SNR_{MFB} = \frac{\mathcal{E}_x||p||^2}{N_0}, \tag{4.4}$$

where $\mathcal{E}_x = \mathbb{E}[||x_k||^2]$ and $N_0 = \mathbb{E}[|z(t)|^2]$.

The main question we first address is, what should the optimum receiver structure be for successive transmission over the channel given in Figure 4.1. In the sequel, we consider $N = 1$ for simplicity. Hence we will drop the subscript from $\varphi_n(t)$ and use it as $\varphi(t)$. Clearly all the discussion can be extended to the case when $N > 1$ complex dimensions.

4.2.1 Matched filter

In this section we derive a set of sufficient statistics for the ISI channel.

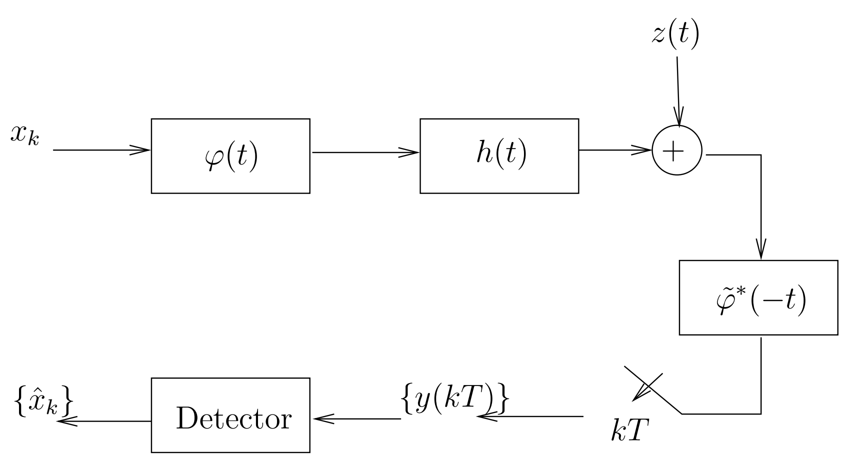
Theorem 4.2.1. *The set of discrete-time samples $\{y(kT)\}_{k\in(-\infty,\infty)}$ where $y(kT) = \tilde{y}(t) * \tilde{\varphi}^*(-t)|_{t=kT}$ are a set of sufficient statistics for the detection of $\{x_k\}$ over a continuous-time ISI channel whose output is $\tilde{y}(t)$,* i.e.,

$$\tilde{y}(t) = h(t) * \sum_n x_n\varphi(t-nT) + \underbrace{z(t)}_{AWGN}$$

and $\tilde{\varphi}(t) \stackrel{def}{=} \frac{h(t)*\varphi(t)}{||h*\varphi||}$

Notes:

1. Here we have assumed $N = 1$, and hence the subscript of φ has been dropped for convenience.
2. The result indicates that the following detector structure incurs no loss of optimality.



Proof: Consider the following problem

$$\tilde{y}(t) = \sum_k x_k ||p|| \tilde{\varphi}(t - kT) + z(t), \;\; t \in (-\infty, \infty)$$

This is like transmitting symbols $\{\breve{x}_k\}$, where $\breve{x}_k = x_k||p||$ using modulating functions $\{\tilde{\varphi}_k(t)\}$ where

$$\tilde{\varphi}_k(t) = \tilde{\varphi}(t - kT) \tag{4.5}$$

We can use a Gram-Schmidt orthogonalization process, for example, to create an *orthonormal* basis for the space spanned by $\{\tilde{\varphi}_k\}$. We can call this orthonormal basis $\{\tilde{\psi}_k\}$ which can be obtained by an *invertible* transformation Γ. That is

$$\begin{aligned} \{\tilde{\psi}_k\} &= \Gamma(\{\tilde{\varphi}_k\}) \\ \{\tilde{\varphi}_k\} &= \Gamma^{-1}(\{\tilde{\psi}_k\}) \end{aligned}$$

Therefore one can write

$$\tilde{y}(t) = \sum_k u_k \tilde{\psi}_k(t) + z(t) \tag{4.6}$$

where $\{u_k\}$ is the coefficients of $\sum_k x_k \tilde{\varphi}_k(t)$ in the new basis. From "single shot" transmission, we know that for *white* Gaussian noise, the optimal receiver projects onto the orthonormal basis directions, *i.e.,* $\{< \tilde{\mathbf{y}}, \tilde{\psi}_k >\}$ form a set of sufficient statistics for detecting $\{u_k\}$.
Now if we apply an invertible transform Γ^{-1} to recover $\{u_k\}$ expressed in their orthogonal basis $\{\varphi_k\}$ (which may not be orthonormal), then we obtain $\{< \tilde{\mathbf{y}}, \tilde{\varphi}_k >\}$ as the set of outputs we process. In other words, $\{u_k\}$ expresses $\tilde{x}(t)$ in terms of the basis $\{\tilde{\varphi}_k\}$ (see (4.6) above), and we can do a change of basis to express $\tilde{x}(t)$ in terms of $\{\tilde{\varphi}_k\}$ which yields the basis coefficients $\{< \tilde{\mathbf{y}}, \tilde{\varphi}_k >\}$. Since, the transform is invertible, by the reversibility theorem we are done.

□

Alternate (more explicit) proof:
Let us consider a finite number of transmissions, *i.e.,* $k = 0, \ldots, K-1$.

$$\tilde{y}(t) = \sum_{k=0}^{K-1} x_k ||p|| \tilde{\varphi}(t - kT) + z(t), \;\; t \in (-\infty, \infty)$$

Optimal receiver projects onto subspace spanned by signal when noise is *white*. That is, let $\tilde{y}_E = \mathcal{P}(\tilde{y})$, where $\mathcal{P}$ projects onto the signal space. Therefore the projections are,

$$< \tilde{y}, \sum x_k ||p|| \tilde{\varphi}_k > = \sum < \tilde{y}, \tilde{\varphi}_k > (x_k ||p||)$$

Hence,

$$y_k = < \tilde{y}, \tilde{\varphi}_k > = \tilde{y}(t) * \tilde{\varphi}_k(-t)|_{t=kT}$$

form a set of sufficient statistics.

4.2.2 Noise whitening

As a result of the matched filtering operation illustrated in Figure 4.2, the noise observed after sampling need not be white. Consider the output of the matched filter which projects the signal onto the basis, *i.e.*,

$$y(kT) = < \tilde{y}, \tilde{\varphi}_k >$$

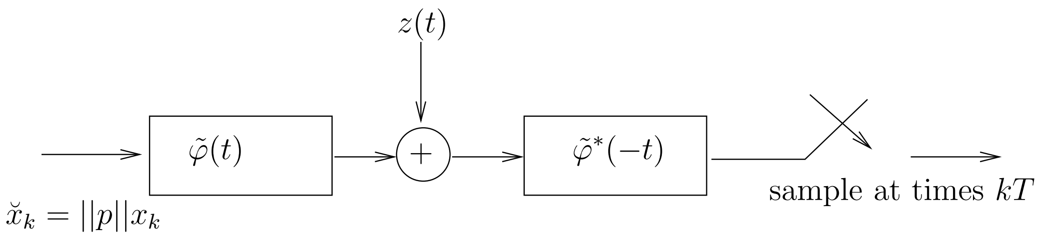


Figure 4.2: Colored noise at the output of the matched filter.

Let us define,

$$y(kT) \stackrel{def}{=} y_k = \sum_n x_n ||p|| < \tilde{\varphi}_n, \tilde{\varphi}_k > + < z, \tilde{\varphi}_k > \tag{4.7}$$

Now,

$$\begin{aligned} < \tilde{\varphi}_n, \tilde{\varphi}_k > &= \int \tilde{\varphi}(t - nT)\tilde{\varphi}^*(t - kT)dt \\ &= \int \tilde{\varphi}(t)\tilde{\varphi}^*(t - (k-n)T)dt \end{aligned}$$

Hence

$$< \tilde{\varphi}_n, \tilde{\varphi}_k > \quad = \quad < \tilde{\varphi}_0, \tilde{\varphi}_{k-n} >$$

Let

$$q_l =< \tilde{\varphi}_0, \tilde{\varphi}_l >$$

Clearly $q_l^* = q_{-l}$ and hence $\{q_l\}$ has conjugate symmetry.
From (4.7),

$$\begin{aligned} y_k &= \sum_n \breve{x}_n q_{k-n} + < \mathbf{z}, \tilde{\varphi}_k > \\ &= \sum_n \breve{x}_{k-n} q_n + < \mathbf{z}, \tilde{\varphi}_k > \end{aligned}$$

Consider the noise process

$$z_n \stackrel{def}{=} < \mathbf{z}, \tilde{\varphi}_n >$$

The spectrum of this noise is,

$$\begin{aligned} E[z_n z_{n-k}^*] &= E\left[\int\int \{z(t)\tilde{\varphi}^*(t-nT)dt\}\{z^*(t)\tilde{\varphi}(\tau-(n-k)T)d\tau\}\right] \\ &= \int_t\int_\tau \tilde{\varphi}^*(t-nT)\tilde{\varphi}(\tau-(n-k)T)E[z(t)z^*(\tau)]dtd\tau \\ &= \int_t\int_\tau \tilde{\varphi}^*(t-nT)\tilde{\varphi}(\tau-(n-k)T)N_0\delta(t-\tau)dtd\tau \\ &= N_0\int_t \tilde{\varphi}^*(t-nT)\tilde{\varphi}(t-(n-k)T)dt \\ &= N_0 < \tilde{\varphi}_{n-k}, \tilde{\varphi}_n >= q_k N_0 \end{aligned}$$

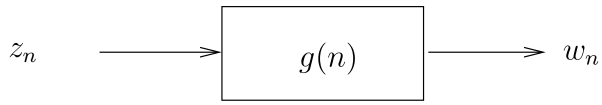


Figure 4.3: The noise whitening filter.

Hence $\{z_k\}$ is not in general white. Therefore, after filtering, we have a colored noise process. Ideally, we would like to work with white noise, and hence we would like to "whiten" it as illustrated in Figure 4.3. To avoid confusion of lots of z's floating around, the D-transform is the Z-transform with $D = z^{-1}$

Definition 4.2.3. *(**The** D**-transform**)*
If $H(z)$ is the z-transform of the sequence $\{h_n\}$, i.e.,

$$H(z) = \sum_n h_n z^{-n},$$

then the D-transform of $\{h_n\}$ is

$$H(D) = \sum_n h_n D^n$$

and we denote the transform pair by

$$\{h_n\} \overset{\mathcal{D}}{\Leftrightarrow} H(D)$$

Now, given the review class, we know that the power spectral density of a random (stationary) process is

$$S_z(D) = \mathcal{D}\{\mathbb{E}[z_k z_{k-n}^*]\}$$

And passing a random process (stationary) through a linear filter $\{g_n\}$ (*i.e.,* with D-transform G(D)) results in

$$S_w(D) = G(D)G^*(\frac{1}{D^*})S_z(D)$$

Now we use the spectral factorization theorem.

Theorem 4.2.2. *(Spectral factorization theorem) A power spectrum $S(D)$, can be factorized into $S(D) = F(D)F^*(\frac{1}{D^*})$ where $F(D)$ is causal, stable and minimum phase (*i.e., *$F(D), F^{-1}(D)$ are analytic for $D < 1$), iff the Paley-Wiener condition holds,* i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\ln[S(e^{j\omega})]|d\omega < \infty$$

We assume, that this holds for the process that we deal with. Now, armed with this we can write

$$S_z(D) = N_0 Q(D) = \breve{F}(D)\breve{F}^*(\frac{1}{D^*})$$

And therefore the filtering $\{z_k\}$ through G(D) gives,

$$S_w(D) = G(D)G^*(\frac{1}{D^*})\breve{F}(D)\breve{F}^*(\frac{1}{D^*})$$

By choosing $G(D) = \frac{\sqrt{N_0}}{\breve{F}^*(\frac{1}{D^*})}$, we see that $S_w(D) = N_0$, *i.e.,* white noise! Now,what happens to our receiver? Note that $S_z(D) = N_0\mathcal{D}\{q_n\}$, hence we have in D-transform notation

$$\begin{aligned} Y(D) &= Q(D) \parallel p \parallel X(D) + Z(D) \\ &= \frac{1}{N_0} S_z(D) \parallel p \parallel X(D) + Z(D) \end{aligned}$$

Therefore,

$$
\begin{aligned}
G(D)Y(D) &= \frac{\sqrt{N_0}}{\breve{F}^*(\frac{1}{D^*})}\frac{\breve{F}(D)\breve{F}^*(\frac{1}{D^*})}{N_0} \parallel p \parallel \breve{X}(D) + W(D) \\
&= \frac{1}{\sqrt{N_0}}\breve{F}(D) \parallel p \parallel X(D) + W(D)
\end{aligned}
$$

So, it is like converting the channel into $\frac{\breve{F}(D)}{\sqrt{N_0}}$. A natural question to ask is whether this procedure gives any advantage.

The spectral factorization can be arranged such that $S_z(D) = \breve{F}(D)\breve{F}^*(\frac{1}{D^*})$ has a *causal* factor $\breve{F}(D)$. Hence we have not only whitened the noise but have converted our channel into a causal sequence. This combination of $\tilde{\varphi}^*(-t)$ and $G(D)$ together is called the ***whitened matched filter***.

Now we have the following problem at hand. Detect sequence $\{\breve{x}_k\}$ when we observe

$$
\tilde{Y}(D) \stackrel{def}{=} G(D)Y(D) = \frac{1}{\sqrt{N_0}}\breve{F}(D) \parallel p \parallel X(D) + W(D) \tag{4.8}
$$

Or in time domain,

$$
\tilde{y}_k = \sum_{n\geq 0} x_{k-n} \underbrace{f_n}_{\mathcal{D}^{-1}(\frac{\|p\|\breve{F}(D)}{\sqrt{N_0}})} + w_k
$$

when w_k is now white Gaussian noise and $\{f_n\}$ is a *causal* sequence. Given the problem stated in (4.8), we would now like to derive efficient detection algorithms while retaining optimality. A naive detection scheme would incur a complexity of M^K where M is the constellation size and K the time window of detection. Clearly we would need better methods than exponential and the next two sections give efficient ML and MAP decoders for this problem.

4.3 Maximum Likelihood Sequence Estimation (MLSE)

Consider again[1],

$$
Y(D) = F(D)X(D) + Z(D) \tag{4.9}
$$

where $F(D) = f_0 + f_1 D + \ldots + f_\nu D^\nu$ *i.e. a finite length response.*
Let

$$
\bar{S}_z(D) = \frac{N_0}{2} \stackrel{def}{=} \sigma^2
$$

Note that the signal to noise ratio (SNR) is given by,

$$
\frac{\mathbb{E}\left[|y(k) - z(k)|^2\right]}{\mathbb{E}\left[|z(k)|^2\right]} = \frac{||f||^2\mathbb{E}\left[|x(k)|^2\right]}{\sigma^2} = \frac{\mathcal{E}_x}{\sigma^2}||f||^2
$$

We make the following assumptions,

- Uniform priors
- $\nu < \infty \Rightarrow$ finite impulse response

Implication: $\{y(k)\}$ is the output of a finite state machine operating over $\mathbb{C}$ (see Figure 4.4), with additive i.i.d. noise process $w(k)$.

[1] We revert back to using $z(k)$ as the notation for noise. In the previous notation this was $w(k)$ and we use $z(k)$ as white noise. Also, we drop the $\tilde{Y}$ notation and use $Y(D)$ instead of $\tilde{Y}(D)$ used in equation (4.8).

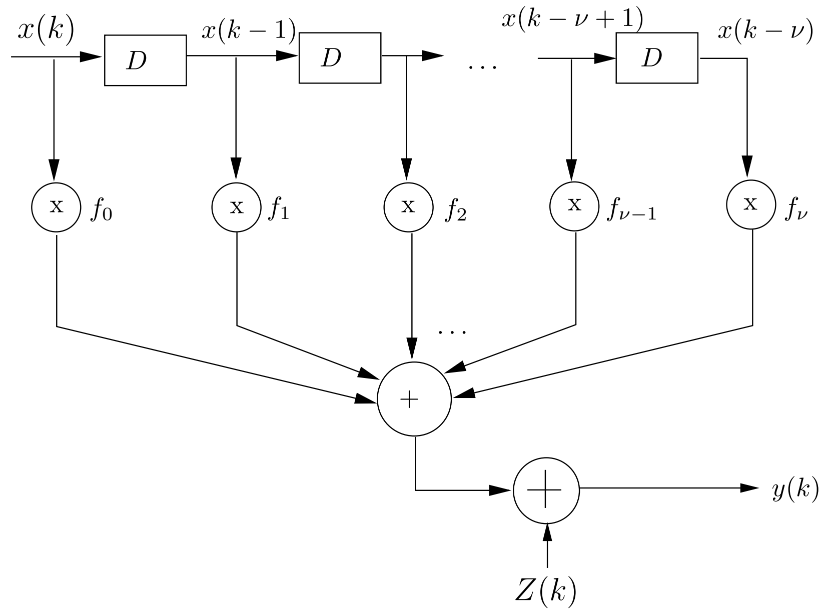


Figure 4.4: Finite state machine representation of output of WMF.

Definition 4.3.1. *We define the state of the finite state machine as* $\mathbf{S}(k)$ *where, let,*

$$\mathbf{s}(k) = \begin{bmatrix} x(k-1) \\ x(k-2) \\ \vdots \\ x(k-\nu) \end{bmatrix} \in \mathbb{C}^\nu$$

Hence $y(k)$ depends only on states $\mathbf{s}(k), \mathbf{s}(k+1)$ and noise $z(k)$.

4.3.1 Viterbi Algorithm

The Maximum Likelihood detection criterion yields,

$$\{\hat{x}(D)\} = \arg\max_{X(D)} \mathbb{P}\{Y(D)|X(D)\}$$

Notes:

- This could deal with sequences that are semi-infinite.
- Since there is a one-to-one map between $\mathbf{s}(D)$ and $X(D)$, we can equivalently pose the problem as estimating the state from noisy observations.
- Now, the form of the problem is similar to the decoding of convolutional codes, *i.e.*, employ the Viterbi algorithm (dynamic programming).
- Let us denote $u(D) = f(D)x(D)$. Hence $u(k)$ depends only on $\mathbf{s}(k)$ and $\mathbf{s}(k+1)$.

Since $z(k)$ is i.i.d. Gaussian, the log-likelihood splits up as

$$\begin{aligned}\log \mathbb{P}\{y(D)|\mathbf{s}(D)\} &= \sum_k \log \mathbb{P}_z\{y(k) - u(k)\} \\ &= \sum_k \left[\log(2\pi\sigma^2) - |y(k) - u(k)|^2\right]\end{aligned}$$

Let us denote

$$\Gamma[s(D)]_{k_1}^{k_2} \stackrel{def}{=} \sum_{k=k_1}^{k_2-1} \log \mathbb{P}_z[y(k) - u(s(k), s(k+1))]$$

$u(\mathbf{s}(k), \mathbf{s}(k+1))$ shows explicit dependence of $u(k)$ on $\mathbf{s}(k), \mathbf{s}(k+1)$.

Dynamic programming principle: Suppose, for the moment, we knew that the state $\mathbf{s}(k)$ at time k was $\mathcal{S}_j$ (one particular choice out of M^ν choices). Then, for any allowable state sequence $\mathbf{s}(D)$, that starts with $\mathbf{s}(0) = \mathbf{0}$ and passes through state $\mathcal{S}_j$ at time k, the log likelihood would break up into two independent parts,

$$\Gamma[s(D)]_0^K = \Gamma[s(D)]_0^k + \Gamma[s(D)]_k^K$$

If $\hat{\mathbf{s}}_j(D)$ be any *allowable* state sequence from time 0 to k, that has maximum log-likelihood $\Gamma[s(D)]_0^k$ among all the allowable state sequences starting from $\mathbf{s}(0) = \mathbf{0}$ and ending at $\mathbf{s}(K) = \mathcal{S}_j$. We call $\hat{\mathbf{s}}_j(D)$, the *survivor* at time k corresponding to state $\mathcal{S}_j$.
Therefore, $\hat{\mathbf{s}}_j(D)$ *must* be the initial segment of the maximum likelihood state sequence, if indeed the true state sequence was $\mathcal{S}_j$ at time k. We do *not* know the state sequence $\mathbf{s}(k)$ but clearly it has to be one of a finite set of sizeM^ν. Consequently, we store survivor sequences for each state and their corresponding log-likelihoods $\Gamma[\hat{s}_j(D)]_0^k$ for $j \in \{1,\ldots, M^\nu\}$.
Thus we store M^ν sequences $\hat{\mathbf{s}}_j(D)$ and their log-likelihoods $\Gamma[\hat{s}_j(D)]_0^k$ and update them as follows:

1. For each of the M allowable extensions of $\hat{\mathbf{s}}_j(D)$ to time $k+1$, compute for each of the M^ν survivors:

$$\Gamma[s_j(D)]_0^{k+1} = \Gamma[\hat{s}_l(D)]_0^k + \ln \mathbb{P}_z[y(k) - u(\mathcal{S}_l, \mathcal{S}_j)] \;\; \forall j \in \{1, \ldots, M^\nu\}$$

   where $l \in \{1,\ldots, M^\nu\}$. This needs $M^\nu M = M^{\nu+1}$ additions.

2. For each state j, compare the log likelihoods $\Gamma[s_j(D)]_0^{k+1}$ of the M extensions that end in state $\mathcal{S}_j$, and select the largest as the corresponding survivor. This needs M^ν, M-ary comparisons.

This summarizes the Viterbi algorithm, which in principle terminates only at $k \to \infty$. However, we can make a decision on a particular state (dynamic programming for ISI channel), if *all* the survivors agree on that state. Moreover in practice, typically good decisions can be made with a finite time horizon. Note that such a sequence estimation can also work if $x(D)$, itself was the result of the finite state machine (*e.g.,* convolutional code). In this case the state machine can be augmented and one can obtain an optimal joint equalization and decoding of the channel code.

4.3.2 Error Analysis

A correct state sequence and an erroneous one are illustrated in Figure 4.5.
Error event: An error event extends from time k_1 to k_2 if the estimated state sequence $\hat{\mathbf{s}}(D)$ is equal to the correct state sequence $s(D)$ at times k_1 and k_2 but nowhere in between. Hence $\hat{\mathbf{s}}(k_1) = \mathbf{s}(k_1), \hat{\mathbf{s}}(k_2) =$

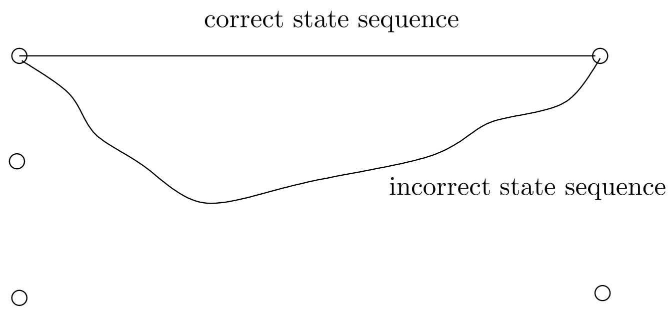


Figure 4.5:

$\mathbf{s}(k_2), \hat{\mathbf{s}}(k) \neq \mathbf{s}(k), k_1 < k < k_2$. The Length of error event is $n = k_2 - k_1 - 1 \geq \nu$.
Clearly for such an error event:

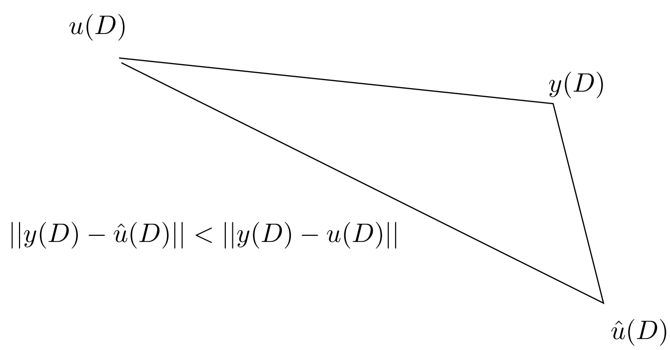
$$\hat{x}(k) = x(k), k_1 - \nu \leq k \leq k_1 - 1 \text{ and } k_2 - \nu \leq k \leq k_2 - 1$$

from the definition of $\mathbf{s}(k)$.
However, $\hat{x}(k_1) \neq x(k_1)$ and $\hat{x}(k_2-\nu-1) \neq x(k_2-\nu-1)$ since, $\hat{\mathbf{s}}(k_1+1) \neq \mathbf{s}(k_1+1)$, and $\hat{\mathbf{s}}(k_2-1) \neq \mathbf{s}(k_2-1)$, we can define,

$$\varepsilon_x(D) \stackrel{def}{=} [x(k_1) - \hat{x}(k_1)] + [x(k_1+1) - \hat{x}(k_1+1)]D + \ldots + [x(k_2 - \nu - 1) - \hat{x}(k_2 - \nu - 1)]D^{n-\nu}$$
$$\varepsilon_u(D) \stackrel{def}{=} [u(k_1) - \hat{u}(k_1)] + [u(k_1+1) - \hat{u}(k_1+1)]D + \ldots + [u(k_2 - 1) - \hat{u}(k_2 - 1)]D^{n}$$

Since $u(D) = f(D)x(D)$, clearly we have $\varepsilon_u(D) = f(D)\varepsilon_x(D)$

Probability of a particular error event:



Think of $u(D)$ being "symbols" in sequence space. Hence, the same way as we did earlier, we can write the NNUB as

$$\mathbb{P}_{e,MLSE} \leq N_e Q\left(\frac{d_{min}}{2\sigma}\right)$$

where d_{min} is the minimum distance between sequences $u(D)$, *i.e.*

$$d_{min} = \min_{u \neq \hat{u}} ||u(D) - \hat{u}(D)||$$

N_e is again $N_e = \sum_x \mathbb{P}_x(\varepsilon_x + \hat{x})N_u$ that is, the average # of nearest neighbors. Note that this is done in sequence space and we calculate over the allowable sequences, *i.e.*, $\varepsilon_x + \hat{x}$ must be an allowable sequence. The main question is to be able to calculate d_{min} for different channels.

4.4 Maximum a-posteriori symbol detection

In section (4.3), we explored the maximum likelihood detector. But if the priors are not uniform the MAP detector is the optimal scheme and this is the focus of the following section.

4.4.1 BCJR Algorithm

For minimizing symbol error probability we use

$$\hat{x}(n) = \arg\max_{x(n)} \mathbb{P}[x(n)|y(0), \dots y(N-1)]$$

The BCJR algorithm efficiently does this computation simultaneously for all $\{\hat{x}(n)\}$.

$$\hat{x}(n) = \arg\max_{x(n)} \mathbb{P}[x(n)|\mathbf{y}_0^{N-1}] = \arg\max_{x(n)} \mathbb{P}[x(n), \mathbf{y}_0^{N-1}]$$

where $\mathbf{y}_{k_1}^{k_2} = \{y(k_1), \dots, y(k_2)\}$

$$\begin{aligned}
\mathbb{P}[x(n), \mathbf{y}_0^{N-1}] &= \sum_{i,j} \mathbb{P}(x(n), \mathbf{y}_0^{n-1}, y(n), \mathbf{y}_{n+1}^{N-1}, \mathbf{s}(n) = \mathcal{S}_i, \mathbf{s}(n+1) = \mathcal{S}_j) \\
&= \sum_{i,j} \mathbb{P}[\mathbf{y}_0^{n-1}, \mathbf{s}(n) = \mathcal{S}_i] \mathbb{P}[x(n), y(n), \mathbf{y}_{n+1}^{N-1}, \mathbf{s}(n+1) = \mathcal{S}_j | \mathbf{y}_0^{n-1}, \mathbf{s}(n) = \mathcal{S}_i] \\
&= \sum_{i,j} \mathbb{P}[\mathbf{y}_0^{n-1}, \mathbf{s}(n) = \mathcal{S}_i] \mathbb{P}[x(n), y(n), \mathbf{s}(n+1) = \mathcal{S}_j | \mathbf{y}_0^{n-1}, \mathbf{s}(n) = \mathcal{S}_i] \\
&\quad \mathbb{P}[\mathbf{y}_{n+1}^{N-1} | x(n), \mathbf{y}_0^{n-1}, y(n), \mathbf{s}(n) = \mathcal{S}_i, \mathbf{s}(n+1) = \mathcal{S}_j] \\
&\overset{(a)}{=} \sum_{i,j} \underbrace{\mathbb{P}[\mathbf{y}_0^{n-1}, \mathbf{s}(n) = \mathcal{S}_i]}_{\alpha_n(i)} \underbrace{\mathbb{P}[x(n), y(n), \mathbf{s}(n+1) = \mathcal{S}_j | \mathbf{s}(n) = \mathcal{S}_i]}_{\gamma_n(i,j)} \\
&\quad \underbrace{\mathbb{P}[\mathbf{y}_{n+1}^{N-1} | \mathbf{s}(n+1) = \mathcal{S}_j]}_{\beta_{n+1}(j)}
\end{aligned}$$

where (a) is due to the Markov property of the finite state machine. Therefore we obtain the relationship,

$$\mathbb{P}[x(n), \mathbf{y}_0^{N-1}] = \sum_{i,j} \alpha_n(i) \gamma_n(i,j) \beta_{n+1}(j)$$

The BCJR algorithm essentially computes $\alpha_n(i)$, and $\beta_{n+1}(j)$ efficiently through the forward-backward recursion.

$$\gamma_n(i,j) = \mathbb{P}[y(n), x(n), \mathbf{s}(n+1)] = \mathcal{S}_j | \mathbf{s}(n) = \mathcal{S}_i]$$

This can be computed for each n, as,

$$\begin{aligned}
\gamma_n(i,j) &= \mathbb{P}[y(n), x(n) | \mathbf{s}(n) = \mathcal{S}_i] \mathbb{P}[\mathbf{s}(n+1) = \mathcal{S}_j | \mathbf{s}(n) = \mathcal{S}_i, x(n), y(n)] \\
&= p(x(n)) p\left(y(n) \mid x(n), \mathbf{s}(n) = \mathcal{S}_i\right) \mathbb{P}[\mathbf{s}(n+1) = \mathcal{S}_j | \mathbf{s}(n) = \mathcal{S}_i, x(n), y(n)]
\end{aligned}$$

If the transition from $\mathcal{S}_i$ to $\mathcal{S}_j$ occurs due to symbol $\tilde{x}(n)$, then

$$\mathbb{P}[\mathbf{s}(n+1) = \mathcal{S}_j | \mathbf{s}(n) = \mathcal{S}_i, x(n), y(n)]$$

is either 1 if $\tilde{x}(n) = x(n)$, and 0 otherwise. Now we need to compute $\{\alpha_n(i)\}$ and $\{\beta_n(j)\}$.

Forward recursion

Computation of $\{\alpha_n(i)\}$

$$\begin{aligned}
\alpha_n(i) &= \mathbb{P}[\mathbf{y}_0^{n-1}, \mathbf{s}(n) = \mathcal{S}_i] \\
&= \sum_k \mathbb{P}[\mathbf{y}_0^{n-1}, \mathbf{s}(n) = \mathcal{S}_i, \mathbf{s}(n-1) = \mathcal{S}_k] \\
&= \sum_k \mathbb{P}[\mathbf{y}_0^{n-2}, y(n-1), \mathbf{s}(n) = \mathcal{S}_i, \mathbf{s}(n-1) = \mathcal{S}_k] \\
&= \sum_k \mathbb{P}[\mathbf{y}_0^{n-2}, \mathbf{s}(n-1) = \mathcal{S}_k]\mathbb{P}[y(n-1), \mathbf{s}(n) = \mathcal{S}_i | \mathbf{y}_0^{n-2}, \mathbf{s}(n-1) = \mathcal{S}_k] \\
&= \sum_k \alpha_{n-1}(k)\gamma_{n-1}(k,i)
\end{aligned}$$

Therefore we get the following forward recursion,

$$\alpha_n(i) = \sum_k \alpha_{n-1}(k)\gamma_{n-1}(k,i). \tag{4.10}$$

Backward recursion

Computation of $\{\beta_n(j)\}$

$$\begin{aligned}
\beta_{n+1}(j) &= \mathbb{P}[\mathbf{y}_{n+1}^{N-1} | \mathbf{s}(n+1) = \mathcal{S}_j] \\
&= \sum_k \mathbb{P}[\mathbf{y}_{n+1}^{N-1}, \mathbf{s}(n+2) = \mathcal{S}_k | \mathbf{s}(n+1) = \mathcal{S}_j] \\
&= \sum_k \mathbb{P}[\mathbf{y}_{n+2}^{N-1}, y(n+1), \mathbf{s}(n+2) = \mathcal{S}_k | \mathbf{s}(n+1) = \mathcal{S}_j] \\
&= \sum_k \mathbb{P}[y(n+1), \mathbf{s}(n+2) = \mathcal{S}_k | \mathbf{s}(n+1) = \mathcal{S}_j] \\
&\quad \mathbb{P}[\mathbf{y}_{n+2}^{N-1} | y(n+1), \mathbf{s}(n+2) = \mathcal{S}_k, \mathbf{s}(n+1) = \mathcal{S}_j] \\
&= \sum_k \gamma_{n+1}(j,k)\beta_{n+2}(k)
\end{aligned}$$

Hence we get the following backward recursion,

$$\beta_{n+1}(j) = \sum_k \gamma_{n+1}(j,k)\beta_{n+2}(k) \tag{4.11}$$

Using the forward-backward recursions, one can then compute the max-a-posteriori symbols $\{\hat{x}(n)\}_{n=0}^{N-1}$.
Notes:

1. The BCJR algorithm computes $\{x(n)\}$ according to minimum *symbol-by-symbol* error probability: This could be different from the sequence of error probability

2. BCJR also gives us "soft-information" on the reliability of the symbols detected.

3. The BCJR algorithm and the Viterbi algorithm (MLSE) applied to same observations (and uniform priors) may give different candidate sequences, since they optimize *different* criteria. The choice of which criteria is important depends on the application.

4.5 Problems

Problem 4.1

Let the transmission over an ISI channel yield after matched filtering the following model,

$$Y(D) = ||p||X(D)Q(D) + Z(D)$$

where $q_l = e^{-2|l|}$ and $S_z(D) = N_0 Q(D)$ and $Q(D)$ is the D-Transform of $\{q_l\}$.
Find the whitening filter $W(D)$ to whiten the noise. Choose the whitening filter such that the resulting communication channel after the whitening filter is causal. That is, $Q(D)W(D)$ is causal.

Problem 4.2

Suppose we are given q_k, (the autocorrelation function of the normalized pulse function) by : $q_0 = 5/4$, $q_1 = q_{-1} = -1/2$, and where the equivalent channel in D-transform resulting out of matching filter is given by :

$$Y(D) = \frac{1}{N_0} S_z(D)||p||X(D) + Z(D)$$

Find $S_z(D)$, $F(D)$ and the resulting channel (write in the temporal domain $G(D)Y(D)$).

Problem 4.3

[Whitening and coloring]

1. Let the transmission over an ISI channel yield after matched filtering the following model,

   $$Y(D) = ||p||X(D)Q(D) + Z(D)$$

   where $q_l = e^{-2|l|}$ and $S_z(D) = N_0 Q(D)$ and $Q(D)$ is the D-Transform of $\{q_l\}$.

   Find the whitening filter $W(D)$ to whiten the noise. Choose the whitening filter such that the resulting communication channel after the whitening filter is causal. That is,

   $$Y(D)W(D) = ||p||X(D) \underbrace{Q(D)W(D)}_{\text{causal channel}} + Z(D)W(D)$$

2. Let $\{Z_n\}$ be a wide-sense stationary random process with $\mathbb{E}Z_n = 0$,

   $$\mathbb{E}Z_n Z_{n-l} = \begin{cases} 1.81 & l = 0 \\ 0.9 & |l| = 1 \\ 0 & \text{else} \end{cases}$$

   and let $\{Z_n\}$ be a real process. Now let $\{U_n\}$ be a white, wide-sense stationary real random process, i.e $\mathbb{E}U_n = 0$, and $\mathbb{E}U_n U_{n-l} = \begin{cases} 1 & l = 0 \\ 0 & \text{else} \end{cases}$

   Find a coloring filter $\{C_n\}$ such that $Z_n = C_n * U_n$

Problem 4.4

An information sequence $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, $X_i \in \{+1, -1\}$ is transmitted over a noisy intersymbol iterference channel. The ith channel output is given by

$$Y_i = \tilde{X}_i + Z_i$$

where Z_i is an i.i.d sequence of Gaussian random variables, and,

$$\tilde{X}_i = \sum_{j=0}^{\infty} X_{i-j} h_j,$$

where h_j is the channel impulse response and is given by

$$h_i = \begin{cases} 1, & \text{if } i \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases}$$

Sketch the state diagram of the finite state machine that produces the ouput sequence $\tilde{\mathbf{X}}$ from the input sequence $\mathbf{X}$. Draw the Viterbi decoding trellis for the received sequence $\mathbf{Y} = [0.28,\ -0.94,\ -0.46, 2.26, 1.52]$, and hence determine the maximum likelihood estimate of $\mathbf{X}$.

Problem 4.5

Considering the channel where $H(D) = 1 + D$, show that d_{min}, the minimum distance between two different paths for the MLSE detector is, in this case, the same as the minimum distance between paths in the shortest possible error event. Assume that channel state is known at the beginning and end of the sequence.

Problem 4.6

Consider transmission over an ISI channel with PAM and symbol period T. Let $\varphi(t) = \frac{1}{\sqrt{T}} \text{sinc}\left(\frac{t}{T}\right)$ and $h(t) = \delta(t) - \frac{1}{2}\delta(t - T)$. Assume that AWGN noise has power spectral density N_0.

1. Determine the pulse response $p(t)$.

2. Determine $||p||$ and $\tilde{\varphi}(t)$.

3. Find the autocorrelation function of the noise after sampling the output of the matched filter. Find the whitening filter such that the resulting channel is causal.

4. Assume that $N_0 = 25/64$, size of PAM is 2 and $x_i \in \{-1, 1\}$. Let the transmitted sequence is $\{1, -1, -1, 1, 1\}$ and the output of the whitened matched filter is $\{0.7, 0.1, -2.0, 0.4, 0.7\}$. Find the maximum likelihood sequence using the Viterbi algorithm. Assume that the initial and last states are 1.

5. Apply the BCJR algorithm when the received sequence is same as in part 4 . Compare the two results and comment.

Problem 4.7

Consider noisy ISI channel given by

$$Y_i = X_i + X_{i-1} + Z_i.$$

where X_i and Y_i are the channel input and output, respectively, at time index i, $\mathbf{Z}$ is a sequence of i.i.d. Gaussian random variables, with zero mean and unit variance and $x_i \in \{-1, 1\}$. Calculate the symbol-wise MAP esitmate of $\mathbf{X}$, using the BCJR algorithm, if the received sequence $\mathbf{Y} = [0.28,\ -0.54,\ -0.46,\ 2.26,\ 1.52]$. You may assume that the channel is in state $+1$ at the begining and the end of the sequence. Compare this to the decoding estimate from the MLSE decoder.

Chapter 5

Equalization: Low complexity suboptimal receivers

In many applications even the efficient implementation of optimal receivers may be too computationally intensive for practical use. In such cases further reductions in complexity are sought at the cost of optimality. This chapter explores several such structures for detection in ISI channels. The chapter starts with a discussion of linear estimation which forms the principles behind all the structures studied in this chapter. We then proceed to derive and analyze several suboptimal schemes for ISI channels using linear estimation as a tool.

5.1 Linear estimation

5.1.1 Orthogonality principle

Suppose

$$\mathbf{y}(k) = \mathcal{A}\mathbf{x}(k) + \mathbf{z}(k)$$

and given observations $\{\mathbf{y}(k)\}$ one wants to estimate $\mathbf{x}(k)$. Here $\mathbf{z}(k)$ is assumed to be Gaussian i.i.d. noise.

Estimation criterion

$$\min \mathbb{E}\left[||\mathbf{x}(k) - \hat{\mathbf{x}}(k)||^2\right]$$

This is called the minimum mean squared error (MMSE) criterion. Our interest is restricted to a class of linear estimators, *i.e.*,

$$\hat{\mathbf{x}}(k) = \mathbf{W}\mathbf{y}(k)$$

where $\mathbf{W}$ is the linear estimator matrix.

Hence

$$\mathbf{e}(k) \stackrel{def}{=} \mathbf{x}(k) - \hat{\mathbf{x}}(k) = \mathbf{x}(k) - \mathbf{W}\mathbf{y}(k)$$

and the problem becomes to find $\mathbf{W}$ such that

$$\mathbf{W}_{opt} = \arg\min_{\mathbf{W}} \mathbb{E}\left[||\mathbf{x}(k) - \mathbf{W}\mathbf{y}(k)||^2\right]$$

and the MMSE is defined as

$$\sigma^2_{MMSE} = \mathbb{E}\left[||\mathbf{x}(k) - \mathbf{W}_{opt}\mathbf{y}(k)||^2\right]$$

Notation: Given a vector (or matrix) $\mathbf{x}$ (or $\mathbf{A}$) we use the notation $\mathbf{x}^*$ (or $\mathbf{A}^*$) to denote Hermitian transpose, *i.e.,* complex conjugation and transpose. We use $\mathbf{x}^t$ (or $\mathbf{A}^t$) to denote ordinary transpose without complex conjugation.
Now, we can write the mean-squared error as,

$$\begin{aligned}\mathbb{E}\left[||\mathbf{x}(k)-\mathbf{W}\mathbf{y}(k)||^2\right] &= trace\{\mathbb{E}\left[[(\mathbf{x}(k)-\mathbf{W}\mathbf{y}(k))(\mathbf{x}(k)-\mathbf{W}\mathbf{y}(k))^*]\right]\} \\ &= trace\{\mathbf{R}_{xx}-\mathbf{R}_{xy}\mathbf{W}^*-\mathbf{W}\mathbf{R}_{yx}+\mathbf{W}\mathbf{R}_{yy}\mathbf{W}^*\} \qquad (5.1)\end{aligned}$$

Now to find $\mathbf{W}_{opt}$, we can differentiate equation (5.1) with respect to $(\mathbf{W})^{th}_{(i,j)}$ element of matrix $\mathbf{W}$. An alternate method would be to "guess" the solution. We use the following basic property of the trace operator for matrices.

Fact 5.1.1. *For matrices $\mathbf{A}\in\mathbb{C}^{m\times n}$ and $\mathbf{B}\in\mathbb{C}^{n\times m}$,*

$$trace(\mathbf{AB}) = trace(\mathbf{BA}).$$

Theorem 5.1.1. *(**Orthogonality principle**) The MSE is minimized if and only if the following condition is met*

$$\mathbb{E}\left[\mathbf{e}(k)\mathbf{y}^*(k)\right]=0 \ \forall k$$

Proof: Let $\widetilde{\mathbf{W}}$ any linear estimator, and $\mathbf{W}_{opt}$ be the linear estimator which satisfies the orthogonality principle.

$$\begin{aligned}\mathbb{E}\left[||\tilde{\mathbf{e}}(k)||^2\right] &= \mathbb{E}\left[||\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)+\mathbf{e}_{opt}(k)||^2\right] \\ &= \mathbb{E}\left[||\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)||^2\right]+\mathbb{E}\left[||\mathbf{e}_{opt}(k)||^2\right]+2\mathrm{Re}\left\{\mathbb{E}\left[\mathbf{e}^*_{opt}(k)\left\{\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)\right\}\right]\right\} \\ &\overset{(a)}{=} \mathbb{E}\left[||\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)||^2\right]+\mathbb{E}\left[||\mathbf{e}_{opt}(k)||^2\right]+2\mathrm{Re}\left\{trace\left[\mathbb{E}\left[\mathbf{e}^*_{opt}(k)\left\{\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)\right\}\right]\right]\right\} \\ &\overset{(b)}{=} \mathbb{E}\left[||\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)||^2\right]+\mathbb{E}\left[||\mathbf{e}_{opt}(k)||^2\right] \\ &\quad +2\mathrm{Re}\left\{trace\left[\mathbb{E}\left[\left\{\mathbf{x}(k)-\widetilde{\mathbf{W}}\mathbf{y}(k)-\mathbf{x}(k)+\mathbf{W}_{opt}\mathbf{y}(k)\right\}\mathbf{e}^*_{opt}(k)\right]\right]\right\} \\ &= \mathbb{E}\left[||\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)||^2\right]+\mathbb{E}\left[||\mathbf{e}_{opt}(k)||^2\right]+2\mathrm{Re}\left\{trace\left[\mathbb{E}\left[(\mathbf{W}_{opt}-\widetilde{\mathbf{W}})\mathbf{y}(k)\mathbf{e}^*_{opt}(k)\right]\right]\right\} \\ &\overset{(c)}{=} \mathbb{E}\left[||\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)||^2\right]+\mathbb{E}\left[||\mathbf{e}_{opt}(k)||^2\right] \\ &\geq \mathbb{E}\left[||\mathbf{e}_{opt}(k)||^2\right]\end{aligned}$$

Where (a) follows because trace of a scalar is the same as the scalar. Also (b) follows since trace is a linear operator *i.e.,*

$$\mathbb{E}\left[trace(\cdot)\right]=trace\{\mathbb{E}\left[(\cdot)\right]\},$$

and because of Fact 5.1.1. Finally (c) follows due to the supposition of the theorem. Therefore we get the if part of the theorem directly due to the inequality $\mathbb{E}\left[||\tilde{\mathbf{e}}(k)||^2\right]\geq\mathbb{E}\left[||\mathbf{e}_{opt}(k)||^2\right]$. We get the only if part by noticing that we need for optimality of *any* other estimator, $\mathbb{E}\left[||\tilde{\mathbf{e}}(k)-\mathbf{e}_{opt}(k)||^2\right]=0$ which means that $\tilde{\mathbf{e}}(k)=\mathbf{e}_{opt}(k)$ almost everywhere, *i.e.,* except over a set of measure zero. This implies that $\widetilde{\mathbf{W}}\mathbf{y}(k)=\mathbf{W}_{opt}\mathbf{y}(k)$ almost everywhere, resulting in $\widetilde{\mathbf{W}}=\mathbf{W}_{opt}$ for optimality. Hence we have proved the theorem.

□

Therefore, using the orthogonality principle, we have

$$\mathbb{E}\left[(\mathbf{x}(k) - \mathbf{W}_{opt}\mathbf{y}(k))\mathbf{y}^*(k)\right] = 0$$

or

$$\mathbf{W}_{opt}\mathbf{R}_{yy} = \mathbf{R}_{xy}$$

where $\mathbf{R}_{yy} = \mathbb{E}\left[\mathbf{y}(k)\mathbf{y}^*(k)\right]$, $\mathbf{R}_{xy} = \mathbb{E}\left[\mathbf{x}(k)\mathbf{y}^*(k)\right]$ and hence

$$\mathbf{W}_{opt} = \mathbf{R}_{xy}\mathbf{R}_{yy}^{-1} \tag{5.2}$$

Now, let us calculate the minimum mean-squared error (MMSE) of the optimal estimator.

$$\begin{aligned}
\sigma^2_{MMSE} &= \mathbb{E}\left[||\mathbf{x}(k) - \mathbf{W}_{opt}\mathbf{y}(k)||^2\right] \\
&= \mathbb{E}\left[(\mathbf{x}(k) - \mathbf{W}_{opt}\mathbf{y}(k))^*\mathbf{e}_{opt}\right] \\
&= \mathbb{E}\left[\mathbf{x}^*(k)\mathbf{e}_{opt}\right] - trace\left\{\mathbb{E}\left[\mathbf{W}_{opt}\mathbf{y}(k)\mathbf{e}_{opt}\right]\right\} \\
&\overset{(a)}{=} \mathbb{E}\left[\mathbf{x}^*(k)(\mathbf{x}(k) - \mathbf{W}_{opt}\mathbf{y}(k))\right] \\
&= \mathbb{E}\left[||\mathbf{x}(k)||^2\right] - trace\mathbb{E}\left[\mathbf{W}_{opt}\mathbf{y}(k)\mathbf{x}^*(k)\right] \\
&= trace\left[\mathbb{E}\left[\mathbf{x}\mathbf{x}^*\right] - \mathbf{W}_{opt}\mathbb{E}\left[\mathbf{y}\mathbf{x}^*\right]\right] \\
&= trace\left[\mathbf{R}_{xx} - \mathbf{W}_{opt}\mathbf{R}_{yx}\right]
\end{aligned}$$

Where (a) follows due to Theorem 5.1.1.

Optimum (non-linear) MMSE estimation

A natural question to ask is what the optimal estimator is, without the restriction to a linear estimator. We next show that the optimal estimator is given by

$$\hat{\mathbf{x}} = \mathbb{E}\left[\mathbf{x} \mid \mathbf{Y} = \mathbf{y}\right].$$

Notes:

1. In general, depending on the distributions $\hat{\mathbf{x}} = \mathbb{E}\left[\mathbf{x}|\mathbf{Y} = \mathbf{y}\right]$ could be a non-linear function of $\mathbf{y}$. For brevity we will denote $\mathbb{E}\left[\mathbf{x}|\mathbf{Y} = \mathbf{y}\right]$ by $\mathbb{E}\left[\mathbf{x}|\mathbf{y}\right]$.

2. If $\mathbf{x}, \mathbf{y}$ are jointly Gaussian , then $\hat{\mathbf{x}} = \mathbb{E}\left[\mathbf{x}|\mathbf{y}\right]$ is linear and hence linear estimators are optimal for estimating Gaussian random processes from another (correlated) Gaussian random process.

3. Here as well we will show (see Theorem 5.1.2), that the orthogonality principle holds, *i.e.*,

   $$(\mathbf{x} - \mathbb{E}[\mathbf{x} \mid \mathbf{y}]) \perp (g(\mathbf{y}))$$

   for any (measurable) function $g(\mathbf{y})$. Hence

   $$\mathbb{E}[e_{opt}\mathbf{y}^*] = 0.$$

Theorem 5.1.2. $\hat{\mathbf{x}}_{opt} = \mathbb{E}[\mathbf{X} \mid \mathbf{Y} = \mathbf{y}]$ *and* $\mathbf{x} - \hat{\mathbf{x}}_{opt} \perp \mathbf{g}(\mathbf{y})$ *for any (measurable) function* $\mathbf{g}$.

Proof: First, it is easy to see that the *tower property* holds, *i.e.,*

$$\mathbb{E}_{\mathbf{x}}[\mathbf{X}] = \mathbb{E}_{\mathbf{y}}\left[\mathbb{E}_{\mathbf{x}|\mathbf{y}}[\mathbf{X}]\right].$$

Hence

$$\begin{aligned}
\mathbb{E}[\mathbf{X}\mathbf{g}^*(\mathbf{y})] &= \mathbb{E}_{\mathbf{y}}\left[\mathbb{E}_{\mathbf{x}|\mathbf{y}}[\mathbf{X}\mathbf{g}^*(\mathbf{y})]\right] = \mathbb{E}_{\mathbf{y}}\left[\left(\mathbb{E}_{\mathbf{x}|\mathbf{y}}[\mathbf{X}]\right)\mathbf{g}^*(\mathbf{y})\right] \\
&= \mathbb{E}_{\mathbf{y}}\left[\left(\mathbb{E}_{\mathbf{x}|\mathbf{y}}[\mathbf{X} \mid \mathbf{Y}]\right)\mathbf{g}^*(\mathbf{y})\right] \\
\Rightarrow \mathbb{E}\left[\left(\mathbf{X} - \mathbb{E}_{\mathbf{x}|\mathbf{y}}[\mathbf{X}]\right)\mathbf{g}^*(\mathbf{y})\right] &= 0.
\end{aligned}$$

Therefore, if we denote $\hat{\mathbf{x}}_{opt} = \mathbb{E}_{\mathbf{x}|\mathbf{y}}[\mathbf{X} \mid \mathbf{Y}]$, then $\mathbf{x} - \hat{\mathbf{x}}_{opt} \perp \mathbf{g}(\mathbf{y})$.
Now, let $\tilde{\mathbf{x}}$ be any other estimator

$$\begin{aligned}
\Rightarrow \mathbb{E}[\| \mathbf{x} - \tilde{\mathbf{x}} \|^2] &= \mathbb{E}[\| \mathbf{x} - \tilde{\mathbf{x}} + \hat{\mathbf{x}}_{opt} - \hat{\mathbf{x}}_{opt} \|^2] \\
&= \mathbb{E}[\| (\hat{\mathbf{x}}_{opt} - \tilde{\mathbf{x}}) + e_{opt} \|^2] \\
&= \mathbb{E}[\| (\hat{\mathbf{x}}_{opt} - \tilde{\mathbf{x}}) \|^2] + \mathbb{E}[\| \mathbf{e}_{opt} \|^2] + 2\mathrm{Re}\left\{ \mathbb{E}[\mathbf{e}_{opt} \underbrace{(\hat{\mathbf{x}}_{opt} - \tilde{\mathbf{x}})^*}_{\mathbf{g}(\mathbf{y})}] \right\} \\
\Rightarrow \mathbb{E}[\| \mathbf{x} - \tilde{\mathbf{x}} \|^2] &= \mathbb{E}[\| (\hat{\mathbf{x}}_{opt} - \tilde{\mathbf{x}}) \|^2] + \mathbb{E}[\| \mathbf{e}_{opt} \|^2] \\
\Rightarrow \tilde{\mathbf{x}} &= \hat{\mathbf{x}}_{opt}, \quad a.e.
\end{aligned}$$

Where the notation *a.e.* means "almost everywhere", *i.e.,* everything except over a set of zero measures.

□

The estimation principles can be applied to a variety of situations. In particular we are interested in applying it to scenarios where we observe a random process $\{\mathbf{y}(k)\}$ which is a noisy observation of the process $\{\mathbf{x}(k)\}$ which we want to estimate. Note that in general the processes $\{\mathbf{x}(k)\}, \{\mathbf{y}(k)\}$ need not be white, *i.e.,* there could be dependencies over time which we can exploit. In this context in the next sections we consider the following cases.

Smoothing: Here we are allowed to use the entire observation $\mathbf{y}(n),\ n \in (-\infty, \infty)$ in order to estimate $\mathbf{x}(k)$. Therefore we can use "future" observations in the estimate of $\mathbf{x}(k)$.

Prediction: In this case we are *only* allowed to process the observations *strictly causally*. Therefore, in estimating $\mathbf{x}(k)$, we can only use observations $\mathbf{y}(n),\ n \in (-\infty, k-1)$. In this problem, we can also formulate a pure prediction problem where the observations $\mathbf{x}(n),\ n \in (-\infty, k-1)$ are used to predict the value of $\mathbf{x}(k)$.

Filtering: In this case we are *only* allowed to process the observations *causally*. Therefore, in estimating $\mathbf{x}(k)$, we can only use observations $\mathbf{y}(n),\ n \in (-\infty, k)$. Therefore the difference between prediction and filtering is that one can use the "current" observation as well for the estimate of $\mathbf{x}(k)$. However, this means that the filtering problem is degenerate if the observations are $\{\mathbf{x}(k)\}$ itself.

5.1.2 Wiener smoothing

Recall, for a random process $\{x(k)\}$, (scalar and wide-sense stationary).
Autocorrelation:

$$\begin{aligned}
r_{xx}(l) &= \mathbb{E}\left[x(k)x^*(k-l)\right] \\
r_{xx}(l) &= r_{xx}^*(-l)
\end{aligned}$$

i.e., $\{r_{xx}(l)\}$ exhibits conjugate symmetry.
Power Spectrum: We define the power spectrum and cross spectrum as,

$$R_{xx}(D) = \mathcal{D}\{E[x_k x_{k-n}^*]\}, \;\; R_{xy}(D) = \mathcal{D}\{E[x_k y_{k-n}^*]\}, \tag{5.3}$$

where $\mathcal{D}(\cdot)$ denotes the D-transform. Note that there is conjugate symmetry in the power spectrum, *i.e.,* $R_{xx}(D) = R_{xx}^*(D^{-*})$.
Notation (mneumonic): Very often in the class we denote the power spectrum and cross spectrum loosely as

$$R_{xx}(D) = \mathbb{E}[X(D)X^*(D^{-*})], \;\; R_{xy}(D) = \mathbb{E}[X(D)Y^*(D^{-*})],$$

but this is an *imprecise* statement. What we really mean is the relationship in (5.3). This notation is really used as a mneumonic to easily write out relationships.
Suppose we observe $\{y(k)\}$ and we want to estimate $\{x(k)\}$ where unlike our previous set-up, there would be correlation between $y(l)$ and $x(k), k \neq l$. Here we would need to "filter" $\{y(k)\}$ to estimate $\{x(k)\}$. We restrict our attention to linear filters, *i.e.,*

$$\hat{x}(k) \;\; = \;\; \sum_n w(n)y(k-n)$$

Hence,

$$\begin{aligned} \hat{X}(D) &= W(D)Y(D) \\ E(D) &= X(D) - \underbrace{W(D)Y(D)}_{\hat{X}(D)} \end{aligned}$$

We use the criterion,

$$\min_{W(D)} \mathbb{E}\left[|e(k)|^2\right]$$

Claim 5.1.1. *The optimal estimator $W_{opt}(D)$ is such that*

$$\mathbb{E}\left[e_{opt}(k)y^*(k-n)\right] = 0, \;\; \forall n$$

where $e_{opt}(k) = x(k) - \hat{x}_{opt}(k)$, where $\hat{X}_{opt}(D) = W_{opt}(D)Y(D)$.

Using this orthogonality principle we have,

$$\mathbb{E}\left[e_{opt}(k)y^*(k-n)\right] = 0, \;\; \forall n \tag{5.4}$$

Hence,

$$\begin{aligned} & \mathbb{E}\left[\left[x(k) - \sum_l w_{opt}(l)y(k-l)\right]y^*(k-n)\right] = 0 \\ \Rightarrow \quad & \mathbb{E}\left[x(k)y^*(k-n)\right] - \sum_l w_{opt}(l)\mathbb{E}\left[y(k-l)y^*(k-n)\right] = 0 \\ \Rightarrow \quad & r_{xy}(n) = \sum_l w_{opt}(l) r_{yy}(n-l) \\ \Rightarrow \quad & R_{xy}(D) = W_{opt}(D)R_{yy}(D) \end{aligned} \tag{5.5}$$

where we have denoted $r_{xy}(n) = \mathbb{E}\left[x(k)y^*(k-n)\right]$ and $r_{yy}(n-l) = \mathbb{E}\left[y(k-l)y^*(k-n)\right]$.

Note: The relationship in (5.4) (seen in more detail in (5.5) above) can be written in mneumonic as

$$\mathbb{E}\left[E(D)Y^*(D^{-*})\right] \quad = \quad 0. \tag{5.6}$$

This is useful since the relationship in (5.5) can be easily seen by simple manipulation as

$$\begin{aligned} & \mathbb{E}\left[\{X(D) - W(D)Y(D)\}\, Y^*(D^{-*})\right] = 0 \\ \Rightarrow \quad & R_{xy}(D) = W_{opt}(D)R_{yy}(D). \end{aligned} \tag{5.7}$$

The relationship given in (5.6) is really again a shorthand for the cross-spectrum relationship as illustrated before in (5.3) and in (5.5). One can represent the orthogonality condition either way as long as the meaning is clear.
Hence

$$W_{opt}(D) = R_{xy}(D)R_{yy}^{-1}(D).$$

is the optimal linear filter to estimate $\{x(k)\}$ from $\{y(k)\}$. As before, the cross-spectrum and the spectrum respectively, are denoted by $R_{xy}(D) = \mathbb{E}[X(D)Y^*(D^{-*})]$, $R_{yy}(D) = \mathbb{E}[Y(D)Y^*(D^{-*})]$.
Note that,

$$\begin{aligned} Y(D) \quad & \Leftrightarrow \quad y(n) \\ \Rightarrow Y(D) \quad & = \quad \sum y(n)D^n \\ \Rightarrow Y^*(D) \quad & = \quad \sum y^*(n)(D^*)^n \\ \Rightarrow Y^*(D^{-1}) \quad & = \quad \sum y^*(n)(D^{-*})^n \\ \Rightarrow Y^*(D^{-*}) \quad & = \quad \sum y^*(n)D^{-n} = \sum y^*(-n)D^n \end{aligned}$$

5.1.3 Linear prediction

Given a sequence $\{x_k\}$, we want to use the past to predict the present using a linear filter, *i.e.*,

$$\hat{x}_k = \sum_{m=1}^{\infty} a_m x_{k-m}$$

Question: Find $\{a_m\}_{m=1}^{\infty}$ such that $\mathbb{E}\left[|x_k - \hat{x}_k|^2\right]$ is minimized.
Using orthogonality condition, we have

$$e_k = (x_k - \hat{x}_k) \perp x_{k-n}, \quad n = 1 \ldots$$

Using this,

$$\mathbb{E}\left[[x_k - \sum_{m=1}^{\infty} a_m x_{k-m}]x_{k-n}^*\right] = 0$$

or,

$$\mathbb{E}\left[x_k x_{k-n}^*\right] = \sum_{m=1}^{\infty} a_m \mathbb{E}\left[x_{k-m}x_{k-n}^*\right]$$

giving,

$$r_x(n) = \sum_{m=1}^{\infty} a_m r_x(n-m), \quad n = 1, \ldots \tag{5.8}$$

Let,

$$g_n = r_x(n) - \sum_{m=1}^{\infty} a_m r_x(n-m) = \sum_{m=0}^{\infty} a'_m r_x(n-m) \tag{5.9}$$

with,

$$a'_0 = 1, a'_m = -a_m, m \geq 1$$

Clearly, due to (5.8) $\{g_n\}$ is a anti-causal sequence. Now, suppose the Paley-Wiener condition holds and we can write the power spectral density (PSD) of $\{x(k)\}$ as,

$$S_x(D) = L(D)L^*(D^{-*})\Gamma_x$$

where $L(D)$ is minimum phase (*i.e.,* zeros and poles of $L(Z)$ are strictly inside the unit circle). Now, taking the D-transform on both sides of (5.9) we get

$$G(D) = A'(D)S_x(D) = A'(D)L(D)L^*(D^{-*})\Gamma_x$$

where $A'(D)$ is causal and monic and G(D) is anti-causal. Hence,

$$\frac{G(D)}{\Gamma_x L^*(D^{-*})} = A'(D)L(D) \tag{5.10}$$

Now, since $L(D)$ is minimum phase, $L^{-1}(D)$ is causal and stable, and $1/L^*(D^{-n})$ is anti-causal [1]. Hence we have LHS of (5.10) which is anti-causal, equal to the RHS which is causal. Also, since $A'(D), L(D)$ are monic and causal, $A'(D)L(D)$ is also monic. Hence for (5.10) to be true, we need

$$A'(D) = \frac{1}{L(D)}$$

Since $L(D)$ is a minimum phase, $A'(D)$ is therefore causal and stable.
Also for $l > 1$

$$\begin{aligned} \mathbb{E}\left[e_k e_{k-l}^*\right] &= \mathbb{E}\left[e_k\{x_{k-l} - \sum_{m=1}^{\infty} a_m x_{k-m-l}\}^*\right] \\ &= \mathbb{E}\left[e_k x_{k-l}^*\right] - \sum_{m=1}^{\infty} a_m^* \mathbb{E}\left[e_k x_{k-m-l}^*\right] \end{aligned}$$

Since due to orthogonality principle, $\mathbb{E}[e_k x_{k-l}^*] = 0$, $\forall l > 1$, we get,

$$\mathbb{E}\left[e_k e_{k-l}^*\right] = 0, l \neq 0$$

i.e., $\{e_k\}$ is a white sequence and the prediction filter is also a whitening filter for the error sequence. The prediction error sequence is called the *innovation sequence* as it conveys the unpredictability or the "new" information in each step of the process.

[1] $L^{-1}(D) = l_0 + \sum_{n=1}^{\infty} l_n D^n \Rightarrow L^{-*}(D^{-*}) = l_0^* + \sum_{n=1}^{\infty} l_n^* D^{-n}$ is anti-causal.

5.1.4 Geometry of random processes

A random variable/vector/process is a mapping from a probability space Ω, to $\mathbb{C}^n$. *i.e.,* if $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} = \mathbf{x}(\omega)$, $X : \Omega \to \mathbb{C}^n$.

Example 5.1.1. *Think of $\omega \in \Omega$ as an outcome of a single trial, and $X(\omega)$ is just a mapping from the outcome of the trial to a complex vector.*

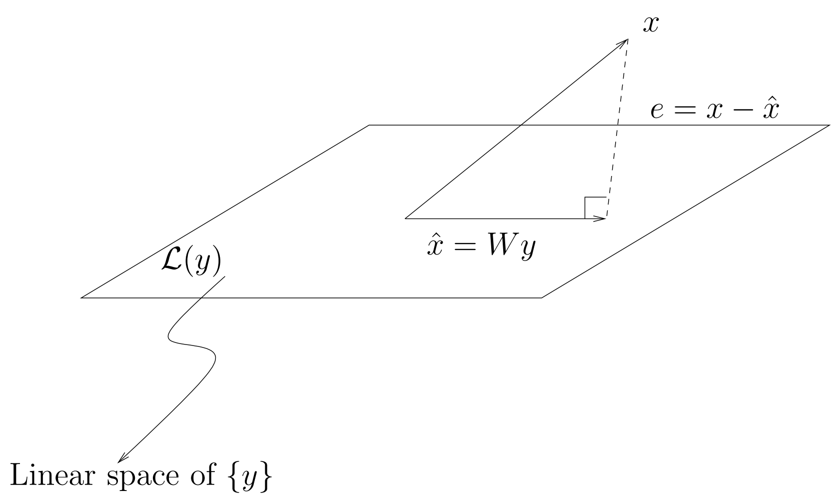


Figure 5.1: Geometry of random processes.

Just as in elementary geometry there is a Pythagorean relationship between $\mathbf{x}$, $\hat{x}_{opt}$ and $\hat{x}$.

Theorem 5.1.3. ***Pythagorean theorem:***

$$\mathbb{E}[\| \mathbf{x} - \hat{\mathbf{x}}_{opt} \|^2] + \mathbb{E}[\| \hat{\mathbf{x}} \|^2] = \mathbb{E}[\| \mathbf{x} \|^2]$$

Inner products: The following properties define an inner product.

1. Linearity: $< \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2, \mathbf{y} >= \alpha_1 < \mathbf{x}_1, \mathbf{y} > + \alpha_2 < \mathbf{x}_2, \mathbf{y} >$
   for α_1, $\alpha_2 \in \mathbb{C}$.
2. Reflexivity: $< \mathbf{x}, \mathbf{y} >=< \mathbf{y}, \mathbf{x} >^*$
3. Non degenerative: $\| \mathbf{x} \|^2 \stackrel{\Delta}{=} < \mathbf{x}, \mathbf{x} >= 0 \Leftrightarrow \mathbf{x} = 0$

Definition 5.1.1. *(Linear Vector Spaces) A linear space $\mathcal{V}$ whose elements are vectors, and a ring of scalars $\mathcal{S}$ through which the operation $\alpha\mathbf{x} \in \mathcal{V}$, for $\alpha \in \mathcal{S}$, $\mathbf{x} \in \mathcal{V}$ is well defined. Moreover the addition operator $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ is also defined.*
$\mathcal{V}$ is a linear space if given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$, $\alpha, \beta \in \mathcal{S}$ the following are satisfied,

(i) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

(ii) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$

(iii) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

(iv) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

(v) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

(vi) $0 \cdot \mathbf{x} = 0,\ 1 \cdot \mathbf{x} = \mathbf{x}$

Usually $\mathcal{S} = \mathbb{C}$ in our application. Also, of importance is $\mathcal{S} = \{$ set of square $n \times n$ matrices $\}$ *i.e.,* $\mathcal{S} = \mathbb{C}^{n\times n}$.
For vector valued random variables $\mathbf{y}, \mathbf{z} \in \mathbb{C}^n$

$$< \mathbf{y}(\omega), \mathbf{z}(\omega) >= \mathbb{E}[\mathbf{y}(\omega)\mathbf{z}^*(\omega)] \in \mathbb{C}^{n\times n}$$

If $\mathcal{S} = \mathbb{C}^{n\times n}$ (*i.e.,* square matrices), then $\mathcal{V}$ is a linear space over $\mathcal{S}$, and is an inner product space since,

$$\mathbb{E}[\mathbf{z}\mathbf{y}^*] = (\mathbb{E}[\mathbf{y}\mathbf{z}^*])^*$$

and hence

$$< \mathbf{z}, \mathbf{y} >=< \mathbf{y}, \mathbf{z} >^*$$

also

$$\| \mathbf{y} \|^2 = \mathbb{E}[\mathbf{y}\mathbf{y}^*] = 0 \Leftrightarrow \mathbf{y} = 0 \ \ a.e.\ (almost\ everywhere)$$

Hence as

$$< \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2, \mathbf{y} >= \alpha < \mathbf{x}_1, \mathbf{y} > + \alpha_2 < \mathbf{x}_2, \mathbf{y} >$$

for $\alpha_1, \alpha_2 \in \mathcal{S}$, clearly the space of vector valued random variables is a linear inner product space. Hence our geometric intuition of orthogonality is precise in this setting.

5.2 Suboptimal detection: Equalization

Let us consider the ISI channel setup introduced in Chapter 4, and illustrated in Figure 5.2. Let us start with just the matched filter output,

$$y_k = \sum_n x_n ||p|| \underbrace{< \tilde{\varphi}_n, \tilde{\varphi}_k >}_{q_{k-n}} + \underbrace{< z, \tilde{\varphi}_k >}_{z_k},$$

which is written compactly as,

$$y_k = \sum_n x_n ||p|| q_{k-n} + z_k$$

or,

$$y_k = ||p|| [x_k * q_k] + z_k \tag{5.11}$$

In the "suboptimal" receivers, we are only interested in minimizing marginal measures, *i.e.* $\mathbb{E}\left[|e_k|^2\right]$. Thus, having colored noise does not change the principles. We could equivalently have worked with the output of the whitened filter (WMF) and done exactly what we will do next. Just for consistency with other notes, we will do it at the output of the matched filter rather than at the WMF.
All these results can be translated to the latter case and we leave this as an exercise for the students.
Basic idea: The output $\{r_k\}$ is to make the equivalent channel as "close" to the AWGN channel as possible. The equalizer is composed of linear filtering blocks and the detection at the output r_x is done "symbol-by-symbol". The complexity of decoding therefore does not grow exponentially with the channel length as in optimal decoding (MLSE, MAPSD). However, the price is paid in terms of performance. We will consider four kinds of structures:

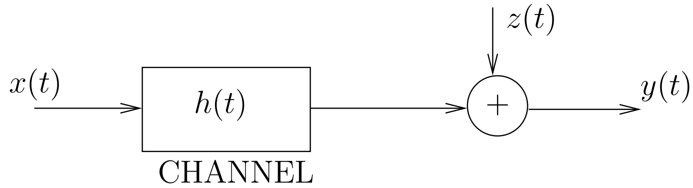


Figure 5.2: The ISI channel.

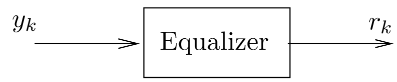


Figure 5.3: Structure of linear equalizer.

1. Zero-forcing equalizer (ZFE): Inverts the channel and eliminate ISI.
2. MMSE linear equalizer (MMSE-LE): Takes noise into account and "inverts" channel as best as possible in the presence of noise.
3. Zero-forcing decision feedback equalizer (ZF-DFE): Uses previous decisions to eliminates ISI and inverts channel.
4. MMSE Decision feedback equalizer (MMSE-DFE): Uses previous decisions to reduce ISI and takes into account presence of noise.

Notation (mneumonic): To re-iterate a point we made earlier about notation. We define the power spectrum and cross spectrum as,

$$S_{xx}(D) = \mathcal{D}\{E[x_k x_{k-n}^*]\}, \;\; S_{xy}(D) = \mathcal{D}\{E[x_k y_{k-n}^*]\}, \tag{5.12}$$

where $\mathcal{D}(\cdot)$ denotes the D-transform.
In this section, very often we denote the power spectrum and cross spectrum loosely as

$$S_{xx}(D) = \mathbb{E}[X(D)X^*(D^{-*})], \;\; S_{xy}(D) = \mathbb{E}[X(D)Y^*(D^{-*})],$$

but this is an *imprecise* statement. What we really mean is the relationship in (5.12). This notation is really used as a mneumonic to easily write out relationships.

5.3 Zero-forcing equalizer (ZFE)

This is the simplest equalizer structure to understand and perhaps analyze. The basic idea is that, if we know that the transmitted symbols have been distorted by a *known* linear filter, then we can eliminate the distortion by just filtering the output through the inverse of the filter. Clearly this does not take into account the presence of additional noise and hence could result in noise enhancement.
Now recall from (5.11) that the output of the matched filter, y_k, is given by,

$$y_k \;\; = \;\; ||p||(x_k * q_k) + z_k$$

Hence in D-transform domain,

$$Y(D) \;\; = \;\; ||p||X(D)Q(D) + Z(D) \tag{5.13}$$

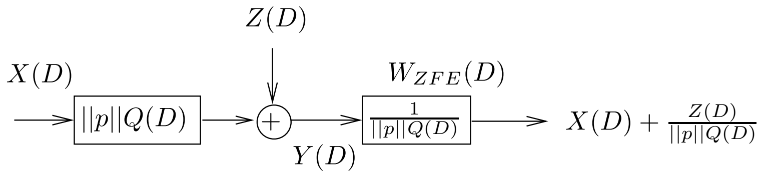


Figure 5.4: Zero-forcing equalizer.

This relationship (5.13) forms the starting point of all the equalization schemes. Now, if we invert the channel using $W_{ZFE}(D)$ given by

$$W_{ZFE}(D) = \frac{1}{\| p \| Q(D)}$$

we get,

$$R(D) \quad = \quad W_{ZFE}(D)Y(D) = \underbrace{\frac{1}{||p||Q(D)}}_{ZFE}[||p||X(D) + Z(D)]$$

Hence we see that,

$$R(D) \quad = \quad X(D) + \frac{Z(D)}{||p||Q(D)}$$

Clearly, the ZFE, $W_{ZFE}(D)$, has eliminated the ISI, but it could have severely enhanced the noise, in that $\frac{1}{Q(D)}$ could actually be very large, and in the absence of noise this would not have mattered. However in the presence of noise, this could severely degrade the SNR.

5.3.1 Performance analysis of the ZFE

We have seen that the noise at the output of the matched filter has a PSD given by,

$$S_z(D) = N_0 Q(D).$$

Normalized per real dimension this is

$$\bar{S}_z(D) = \frac{N_0}{2} Q(D).$$

Now, after the ZFE,

$$r_k = x_k + z_k^{ZFE}$$

where,

$$z_k^{ZFE} = \mathcal{D}^{-1}\left[\frac{Z(D)}{||p||Q(D)}\right].$$

Hence, per dimension, the PSD of the noise z_k^{ZFE} is,

$$\bar{S}_{z^{ZFE}}(D) = \frac{N_0}{2} Q(D) \frac{1}{||p||Q(D)} \frac{1}{||p||Q^*(D^{-*})}.$$

But

$$Q(D) = Q^*(D^{-*})$$

due to conjugate symmetry of $\{q_l\}$.
Therefore,

$$\bar{S}_{z^{ZFE}}(D) = \frac{N_0}{2}\frac{1}{||p||^2}\frac{1}{Q(D)} = \frac{N_0}{2}\frac{W_{ZFE}(D)}{||p||}.$$

Now,

$$\begin{aligned}
\bar{\sigma}^2_{ZFE} &= \frac{T}{2\pi}\int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} \bar{S}_{z^{ZFE}}(e^{-j\omega T})d\omega \\
&= \frac{T}{2\pi}\int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} \frac{N_0}{2}\frac{W_{ZFE}(e^{-j\omega T})}{||p||}d\omega \\
&= \frac{N_0}{2}\frac{1}{||p||}\underbrace{\left[\frac{T}{2\pi}\int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} W_{ZFE}(e^{-j\omega T})d\omega\right]}_{w_{ZFE(0)}}
\end{aligned}$$

Thus we have

$$\begin{aligned}
SNR_{ZFE} &= \frac{\bar{\mathcal{E}}_x}{\bar{\sigma}^2_{ZFE}} \\
&= \frac{\bar{\mathcal{E}}_x}{\frac{N_0}{2}\frac{1}{||p||}w_{ZFE(0)}} \\
&= \left(\frac{\bar{\mathcal{E}}_x||p||}{\frac{N_0}{2}}\right)\frac{1}{w_{ZFE(0)}}
\end{aligned}$$

Noise enhancement: The basic problem occurs when $Q(D)$ has zeroes close to the unit circle as seen in Figure 5.3.1. Hence, inverting $Q(D)$ results in a "gain" that becomes large and hence enhances the

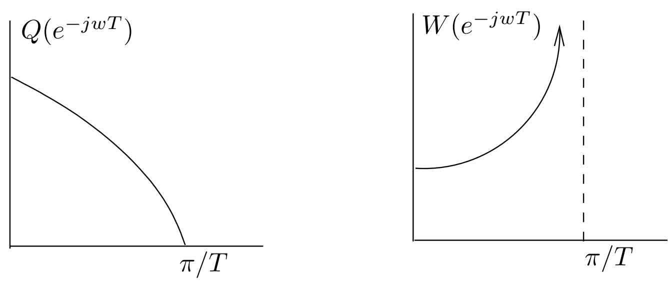


Figure 5.5: The noise enhancement in ZFE.

noise power that was ignored.

5.4 Minimum mean squared error linear equalization (MMSE-LE)

In order to deal with the noise enhancement of the ZFE, we need to take into account the presence of noise. The first method is to modify the ZFE to take the noise into account, by finding a linear filter that minimizes the output noise variance.

$$e_k = x_k - w_k * y_k \tag{5.14}$$

The MMSE-LE, attempts to find the linear filter according to the MMSE criterion applied to (5.14) as,

$$W_{MMSE-LE}(D) = \arg\min_{W(D)} \mathbb{E}\left[|e_k|^2\right]$$

Now, we use the MMSE machinery we have developed in Section 5.1. We want to find $W_{MMSE-LE}(D)$ such that MMSE of output is minimized. By using the orthogonality principle[2] as in (5.5), we have,

$$\mathbb{E}\left[E(D)Y^*(D^{-*})\right] = 0$$

Hence we get,

$$\mathbb{E}\left[(X(D) - W_{MMSE-LE}(D)Y(D))(Y^*(D^{-*}))\right] = 0$$

This gives us,

$$S_{xy}(D) = W_{MMSE-LE}(D)S_{yy}(D),$$

or

$$W_{MMSE-LE}(D) = \frac{S_{xy}(D)}{S_{yy}(D)} = \frac{||p||Q(D)\mathcal{E}_x}{||p||^2 Q^2(D)\mathcal{E}_x + N_0 Q(D)},$$

giving us

$$W_{MMSE-LE}(D) = \frac{||p||\mathcal{E}_x}{||p||^2 Q(D)\mathcal{E}_x + N_0} = \frac{1}{||p||Q(D) + \left(\frac{N_0}{||p||\mathcal{E}_x}\right)} \tag{5.15}$$

Note that the difference between $W_{MMSE-LE}(D)$ and $W_{ZFE}(D)$ is the additive term in the denominator of (5.15). Even if $Q(e^{-j\omega T}) = 0$ for some ω, the gain of the $W_{MMSE-LE}(D)$ does not blow up at these frequencies, as would $W_{ZFE}(D)$. Note that as $\frac{N_0}{\mathcal{E}_x} \to 0$ or SNR $\to \infty$, the MMSE-LE tends towards the ZFE as one would expect.

5.4.1 Performance of the MMSE-LE

At the output of the MMSE-LE, the D-transform is,

$$\begin{aligned} R(D) &= W_{MMSE-LE}(D)Y(D) \\ &= \frac{1}{||p||\left(Q(D) + \frac{1}{SNR_{MFB}}\right)} X(D)Q(D)||p|| + \frac{Z(D)}{||p||\left(Q(D) + \frac{1}{SNR_{MFB}}\right)} \end{aligned}$$

where, as defined in definition 4.2.2,

$$SNR_{MFB} = \frac{\mathcal{E}_x ||p||^2}{N_0} = \frac{\bar{\mathcal{E}}_x ||p||^2}{N_0/2} \tag{5.16}$$

[2] Again recall that the precise relationship is given in (5.5), but we use the mneumonic given in (5.6) for convenience of manupulation.

Therefore

$$\begin{aligned} R(D) &= \left[\frac{Q(D)}{Q(D)+1/SNR_{MFB}}\right] X(D) + \underbrace{\left[\frac{1}{\|p\|\left(Q(D)+1/SNR_{MFB}\right)}\right] Z(D)}_{Z'(D)} \\ &= \underbrace{\left\{1-\frac{1/SNR_{MFB}}{Q(D)+1/SNR_{MFB}}\right\}}_{1-V(D)} X(D) + Z'(D) \end{aligned}$$

Where we have defined,

$$V(D) = \frac{\|p\|}{SNR_{MFB}} W_{MMSE-LE}(D) = \frac{1}{SNR_{MFB}\left[Q(D)+\frac{1}{SNR_{MFB}}\right]} \tag{5.17}$$

Writing it in time domain, we get

$$r_k = x_k - v_k * x_k + z'_k$$

Let us examine, the contribution of x_k in $(v_k * x_k)$,

$$v_k * x_k = \sum_n v_n x_{k-n} = v_0 x_k + \sum_{n\neq 0} v_n x_{k-n}$$

where the second term does not depend on x_k.
Hence we have

$$r_k = x_k - v_0 x_k - \underbrace{\sum_{n\neq 0} v_n x_{k-n} + z'_k}_{e'_k}$$

yielding,

$$r_k = (1-v_0)x_k + e'_k \tag{5.18}$$

Hence one would naturally have defined the detection SNR as,

$$SNR_{MMSE-LE,U} = \frac{\mathcal{E}_x (1-v_0)^2}{\mathbb{E}\left[|e'_k|^2\right]} \tag{5.19}$$

However, in the MMSE minimization, one computes $\mathbb{E}\left[|e_k|^2\right]$, where

$$e_k \quad = x_k - w_{MMSE-LE}(k) * y_k = x_k - r_k$$

Using this in (5.18),

$$e_k \quad = v_0 x_k - e'_k$$

Hence $\frac{\mathcal{E}_x}{\mathbb{E}[|e_k|^2]}$ is *not* the same as what a detector encounters in (5.18). Therefore, the discrepancy needs to be taken into account in the analysis of the MMSE-LE. Let us first calculate,

$$\sigma^2_{MMSE-LE} = \mathbb{E}\left[|e_k|^2\right]$$

Similar to the ZFE analysis,

$$\begin{aligned} S_{EE}(D) &= \mathcal{D}\left\{\mathbb{E}\left[e_l e^*_{l-k}\right]\right\} \\ &= \mathcal{E}_x - W^*_{MMSE-LE}(D^{-*}) S_{XY}(D) - W_{MMSE-LE}(D) S^*_{XY}(D^{-*}) \\ &\quad + W_{MMSE-LE}(D) S_{YY}(D) W^*_{MMSE-LE}(D^{-*}) \\ &= \mathcal{E}_x - W_{MMSE-LE}(D) S_{YY}(D) W^*_{MMSE-LE}(D^{-*}) \end{aligned} \tag{5.20}$$

Now,

$$W_{MMSE-LE}(D) = \frac{||p||Q(D)\mathcal{E}_x}{||p||^2 Q^2(D) + N_0 Q(D)} = \frac{1}{||p||\left[Q(D) + 1/SNR_{MFB}\right]} \tag{5.21}$$

and

$$S_{YY}(D) = ||p||^2 Q^2(D)\mathcal{E}_x + N_0 Q(D) = \mathcal{E}_x ||p||^2 Q(D)\left[Q(D) + \frac{1}{SNR_{MFB}}\right] \tag{5.22}$$

Inserting (5.21) and (5.22) into (5.20) we obtain

$$\begin{aligned} S_{EE}(D) &= \mathcal{E}_x - \frac{\mathcal{E}_x Q(D)}{\left(Q(D) + \frac{1}{SNR_{MFB}}\right)} \\ &= \frac{\mathcal{E}_x / SNR_{MFB}}{Q(D) + \frac{1}{SNR_{MFB}}} \\ &= \frac{\frac{N_0}{||p||}}{||p||\left[Q(D) + \frac{1}{SNR_{MFB}}\right]} \end{aligned} \tag{5.23}$$

Now using (5.16) and (5.17) in (5.23), we get

$$S_{EE}(D) = \mathcal{E}_x V(D) = \frac{N_0}{||p||} W_{MMSE-LE}(D)$$

Hence

$$\mathbb{E}\left[|e_k|^2\right] = \sigma^2_{MMSE-LE} = \mathcal{E}_x v_0 = \frac{N_0}{||p||} w_{MMSE-LE}(0). \tag{5.24}$$

But the performance really depends upon $\mathbb{E}\left[|e'_k|^2\right]$ (see (5.18)). Now we relate them as,

$$\mathbb{E}\left[|e_k|^2\right] = v_0^2 \mathcal{E}_x + \mathbb{E}\left[|e'_k|^2\right]$$

Therefore,

$$\mathbb{E}\left[|e'_k|^2\right] = \mathbb{E}\left[|e_k|^2\right] - v_0^2 \mathcal{E}_x = \mathcal{E}_x v_0 - v_0^2 \mathcal{E}_x \tag{5.25}$$

Thus $\sigma^2_{MMSE-LE,U} = \mathbb{E}[|\, e'_k \,|^2]$ is,

$$\sigma^2_{MMSE-LE,U} = \mathcal{E}_x v_0 [1 - v_0] \tag{5.26}$$

Inserting (5.26) in (5.19) we get,

$$\begin{aligned} SNR_{MMSE-LE} &= \frac{\mathcal{E}_x}{\sigma^2_{MMSE-LE}} = \frac{\mathcal{E}_x}{\mathcal{E}_x v_0} = \frac{1}{v_0} \\ SNR_{MMSE-LE,U} &= \frac{\mathcal{E}_x (1 - v_0)^2}{\mathcal{E}_x v_0 (1 - v_0)} = \frac{1 - v_0}{v_0} = \frac{1}{v_0} - 1 \end{aligned}$$

Hence,

$$SNR_{MMSE-LE} = 1 + SNR_{MMSE-LE,U} \tag{5.27}$$

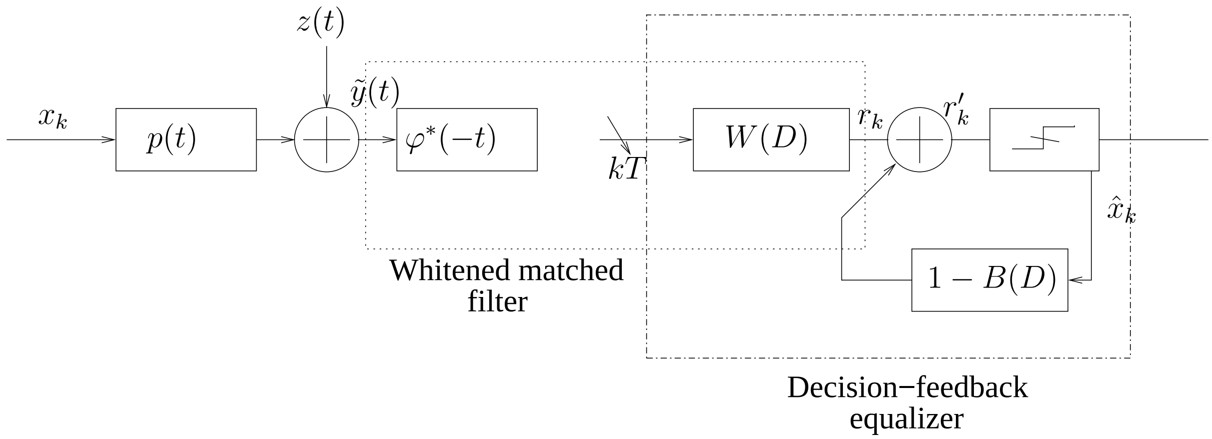


Figure 5.6: The decision feedback equalizer.

The "biased" SNR *i.e.,* $SNR_{MMSE-LE}$ arises directly out of the calculation deriving that MMSE-LE. However, by looking at (5.18), one realizes that the detector actually works with a different SNR, *i.e.,* $SNR_{MMSE-LE,U}$ and hence we need to be careful in finding the performance of the MMSE-LE to use the correct "noise" variance.

Till now we have considered two flavors of the equalizer structure. Both the ZFE and the MMSE-LE attempts to filter the received sequence through a linear filter ($W_{ZFE}(D)$ and $W_{MMSE-LE}(D)$ respectively) in order to convert the ISI problem to be "close" to an AWGN problem. The ZFE did that by inverting the channel, but could cause noise enhancement. The MMSE-LE took the noise into account, but it now has transmitted symbols as part of the "noise". Can we take advantage of the fact that the noise contains some past/future transmitted symbols? This idea forms the basis of the decision feedback equalizer, which we study next.

5.5 Decision-feedback equalizer

The basic idea of decision feedback equalization comes from the realization that one could potentially use previous decisions while attempting to estimate the current symbol.
The derivation and analysis of DFE requires us to make a strong assumption, that the decisions are indeed correct! Without this, the analysis of the DFE is still an open question. We will make this assumption and proceed with the DFE.
Criterion for the MMSE-DFE: The basic optimization problem is,

$$\min_{W(D),B(D),b_0=1} \mathbb{E}\left[|x_k - r'_k|^2\right]$$

Notes:

1. In order to utilize past decisions, one should ensure that r_k depends *only* on the past symbols.

2. The feedforward filter $W(D)$ shapes the sequence$\{y_k\}$ in order to have only "trailing" ISI terms.

3. We have seen one structure that caused the equivalent channel to be causal *i.e.,* the whitened matched filter.

4. We need $B(D)$ such that, $B(D)$ is causal and monic, *i.e.,*

$$B(D) = 1 + b_1 D + b_2 D^2 + \dots$$

which implies that

$$1 - B(D) = -\sum_{n=1}^{\infty} b_n D^n$$

and hence $X(D)[1 - B(D)]$ depends only on past decisions as shown in Figure 5.6.

Main steps in deriving the MMSE-DFE:

Step 1 Fix feedback filter $B(D)$ and find feedforward filter $W(D)$, in terms of $B(D)$, such that $\mathbb{E}\left[|(x_k - r'_k)|^2\right]$ is minimized.

Step 2 Express result of operating $W(D)$ on $Y(D)$, *i.e.*, express $W(D)$ in terms of $B(D)$ and set up linear prediction problem.

Step 3 Solve linear prediction problem to find the causal, $B(D)$ that minimizes $\mathbb{E}\left[|(x_k - r'_k)|^2\right]$. And hence find the MMSE-DFE, *i.e.,* $\{W(D), B(D)\}$.

Step 4 *(Analysis) As in the MMSE-LE, remove the bias term to find the equivalent* $SNR_{MMSE-DFE,U}$.

We will first focus on step 1-3 to obtain the MMSE-DFE and then analyze its performance through step 4.

Step # 1:

Let us fix $B(D)$, and then find $W(D)$ that minimizes MMSE criterion. We can write the error $E(D)$ as,

$$E(D) = X(D) - \underbrace{\{W(D)Y(D) + (1 - B(D))X(D)\}}_{R'(D)}$$

Hence,

$$E(D) = B(D)X(D) - W(D)Y(D) \tag{5.28}$$

Now, to find $W(D)$ that minimizes $\mathbb{E}\left[|e_k|^2\right]$, we use the orthogonality principle[3] (see (5.5)) as,

$$\begin{aligned} \mathbb{E}\left[E(D)Y^*(D^{-*})\right] &= \mathbb{E}\left[[B(D)X(D) - W(D)Y(D)]Y^*(D^{-*})\right] = 0 \\ &= B(D)S_{XY}(D) - W(D)S_{YY}(D) = 0 \end{aligned}$$

Hence,

$$W(D) = B(D)\frac{S_{XY}(D)}{S_{YY}(D)} = B(D)W_{MMSE-LE}(D) = \frac{B(D)}{||p||\left(Q(D) + \frac{1}{SNR_{MFB}}\right)} \tag{5.29}$$

Step # 2: Now, we express the resulting error in terms of $B(D)$ by substituting (5.29) in (5.28) as,

$$\begin{aligned} E(D) &= B(D)X(D) - B(D)W_{MMSE-LE}(D)Y(D) \\ &= B(D)\underbrace{[X(D) - W_{MMSE-LE}(D)Y(D)]}_{U(D)} \\ &= (1 + B(D) - 1)U(D) = U(D) - [1 - B(D)]U(D) \end{aligned}$$

[3] Again recall that the precise relationship is given in (5.5), but we use the mneumonic given in (5.6) for convenience of manupulation.

where $1 - B(D)$ is a strictly causal sequence. Now, this is exactly like the linear prediction problem that we studied in Section 5.1.3, where we want to predict the sequence $\{u_k\} = \mathcal{D}^{-1}(U(D))$.
Step # 3: In Section 5.1.3, we studied the linear prediction problem and we found the optimal linear predictor as,

$$B_{opt}(D) = \frac{1}{L(D)}$$

where

$$S_U(D) = \Gamma_U L(D) L^*(D^{-*})$$

through spectral factorization. Now let us find $S_U(D)$.

$$\begin{aligned} S_U(D) &= \mathbb{E}\left[[X(D) - W_{MMSE-LE}(D)Y(D)][X^*(D^{-*}) - W^*_{MMSE-LE}(D^{-*})Y^*(D^{-*})]\right] \\ &\overset{(a)}{=} S_{XX}(D) - W^*_{MMSE-LE}(D^{-*})S_{YY}(D)W_{MMSE-LE}(D) \end{aligned}$$

where (a) follows because $W_{MMSE-LE}(D) = \frac{S_{XY}(D)}{S_{YY}(D)}$.
Hence,

$$\begin{aligned} S_U(D) &= \mathcal{E}_x - \frac{\mathcal{E}_x Q(D)}{Q(D) + \frac{1}{SNR_{MFB}}} \\ &= \frac{N_0/||p||^2}{Q(D) + \frac{1}{SNR_{MFB}}} \\ &\overset{(b)}{=} \Gamma_U L(D) L^*(D^{-*}) \end{aligned}$$

where (b) is the spectral factorization, with $L(D)$ being causal, monic and minimum phase.
Then we get

$$B_{opt}(D) = \frac{1}{L(D)}$$

which is still causal, monic and stable. Hence

$$\begin{aligned} S_E(D) &= B(D) S_U(D) B^*(D^{-*}) \\ &= \frac{1}{L(D)} \Gamma_U L(D) L^*(D^{-*}) \frac{1}{L^*(D^{-*})} \\ &= \Gamma_U \end{aligned}$$

Alternatively, if

$$Q(D) + \frac{1}{SNR_{MFB}} = \gamma_0 G(D) G^*(D^{-*}) \tag{5.30}$$

then

$$S_U(D) = \frac{N_0/||p||^2}{Q(D) + \frac{1}{SNR_{MFB}}} = \frac{N_0/||p||^2}{\gamma_0 G(D) G^*(D^{-*})}$$

Thus in this notation, $\Gamma_U = \frac{N_0}{||p||^2 \gamma_0}$, $L(D) = \frac{1}{G(D)}$ then

$$B_{opt}(D) = G(D). \tag{5.31}$$

Now, let us substitute this in (5.29) to get,

$$\begin{aligned} W_{opt}(D) &= B_{opt}(D)W_{MMSE-LE}(D) \\ &= G(D)\frac{1}{\|p\|\left(Q(D)+\frac{1}{SNR_{MFB}}\right)} \\ &= G(D)\frac{1}{\|p\|\gamma_0 G(D)G^*(D^{-*})} \\ &= \frac{1}{\|p\|\gamma_0}\frac{1}{G^*(D^{-*})} \end{aligned} \tag{5.32}$$

So, the main computation is to figure out the spectral factorization of $\left[Q(D)+\frac{1}{SNR_{MFB}}\right]$. For the choice of $B_{opt}(D)$ and $W_{opt}(D)$, the error spectrum is:

$$\begin{aligned} S_{EE}(D) &= B_{opt}(D)S_U(D)B^*(D^{-*}) \\ &= \Gamma_U = \frac{N_0}{\gamma_0\|p\|^2} \end{aligned} \tag{5.33}$$

5.5.1 Performance analysis of the MMSE-DFE

First the PSD of the error sequence is,

$$S_{EE}(D) = \frac{N_0}{\gamma_0\|p\|^2},$$

hence the per-dimension PSD is

$$\bar{S}_{EE}(D) = \frac{N_0/2}{\gamma_0\|p\|^2}$$

Thus, the error sequence of the MMSE-DFE (with *no* decision error propagation) is white, when it is minimized.
Now, to compute γ_0 we illustrate this with the following. Let us assume that we have a rational spectrum $S(D)$, *i.e.*,

$$S(D) = \gamma_0\frac{\prod_{k=1}^{M}(1-c_kD)(1-c_k^*D^{-1})}{\prod_{k=1}^{N}(1-d_kD)(1-d_k^*D^{-1})} \tag{5.34}$$

Claim 5.5.1. *For $S(D)$ of the form given in (5.34)*

$$\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln S(e^{-j\omega T})d\omega = \ln\gamma_0$$

if the finite energy constraint is satisfied, i.e.,

$$R(0) = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}S(e^{-j\omega T})d\omega < \infty$$

Notes:

1. This result actually holds in more generality than just rational spectrum, but we do the proof under this restrictive assumption.
2. The finite energy constraint for rational spectra is equivalent to saying that there are no poles of $S(D)$ on the unit circle.

Proof:

$$\begin{aligned}\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln S(e^{j\omega T})d\omega &= \ln\gamma_0+\sum_{k=1}^{M}\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln(1-c_k e^{-j\omega T})d\omega\\ &+ \sum_{k=1}^{M}\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln(1-c_k^* e^{j\omega T})d\omega\\ &- \sum_{k=1}^{N}\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln(1-d_k e^{-j\omega T})d\omega\\ &- \sum_{k=1}^{N}\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln(1-d_k^* e^{j\omega T})d\omega \qquad (5.35)\end{aligned}$$

Now, if $a \in \mathbb{C}$, $|a| \le 1$, $a = |a|e^{j\phi_a}$

$$\begin{aligned}\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln[1-ae^{-j\omega T}][1-a^*e^{j\omega T}]d\omega &= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln[1+|a|^2-2|a|\cos(\omega T-\phi_a)]d\omega\\ &\overset{(a)}{=} 0\end{aligned}$$

where (a) follows from standard integral tables.
Inserting this in (5.35) we get

$$\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln S(e^{jwT})dw = \ln\gamma_0$$

This formula is called the Szego formula.

□

Using the Szego formula for the MMSE-DFE, *i.e.,* using it in (5.33) we obtain

$$\sigma^2_{MMSE-DFE} = \frac{N_0/2}{||p||^2}\exp[-\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln\left(Q(e^{-j\omega T})+\frac{1}{SNR_{MFB}}\right)d\omega]$$

This is called Salz formula as it was derived by Salz in 1973. Hence we get,

$$\begin{aligned}SNR_{MMSE-DFE} &= \frac{\mathcal{E}_x}{\sigma^2_{MMSE-DFE}}\\ &= SNR_{MFB}\exp[\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\ln\left(Q(e^{-j\omega T})+\frac{1}{SNR_{MFB}}\right)d\omega]\\ &= \gamma_0 SNR_{MFB} = \frac{\gamma_0||p||^2\mathcal{E}_x}{N_0}\end{aligned}$$

Now, we have issue of "bias", *i.e.,* to check whether this $SNR_{MMSE-DFE}$ is the right quantity to examine[4]. Let us first find out if we have a biased receiver.

$$\begin{aligned}R'(D) &= R(D)+[1-B(D)]X(D)\\ &= W(D)Y(D)+X(D)-B(D)X(D)\\ &= X(D)-[B(D)X(D)-W(D)Y(D)]\\ &= X(D)-E(D)\end{aligned}$$

[4]This was the same issue investigated in the MMSE-LE.

The receiver makes decisions on x_k, based on r'_k

$$
\begin{aligned}
R'(D) &= X(D) - \left[G(D)X(D) - \frac{1}{||p||\gamma_0} \frac{Y(D)}{G^*(D^{-*})} \right] \\
&= X(D) - \left[G(D)X(D) - \frac{1}{||p||\gamma_0} \frac{||p||Q(D)}{G^*(D^{-*})} X(D) - \frac{1}{||p||\gamma_0} \frac{Z(D)}{G^*(D^{-*})} \right] \\
&= \left\{ 1 - G(D) + \frac{Q(D)}{\gamma_0 G^*(D^{-*})} \right\} X(D) + \frac{1}{||p||\gamma_0} \frac{1}{G^*(D^{-*})} Z(D) \\
&= \left\{ 1 - \frac{-Q(D) + \gamma_0 G(D) G^*(D^{-*})}{\gamma_0 G^*(D^{-*})} \right\} X(D) + \frac{1}{||p||\gamma_0} \frac{1}{G^*(D^{-*})} Z(D) \\
&\overset{(a)}{=} \underbrace{\left\{ 1 - \frac{\frac{1}{SNR_{MFB}}}{\gamma_0 G^*(D^{-*})} \right\}}_{1-V(D)} X(D) + \underbrace{\frac{1}{||p||\gamma_0 G^*(D^{-*})} Z(D)}_{Z'(D)}
\end{aligned}
$$

Where (a) follows due to (5.30) and we have defined

$$
V(D) = \frac{\frac{1}{SNR_{MFB}}}{\gamma_0 G^*(D^{-*})} \tag{5.36}
$$

Note that $V(D) = \frac{\frac{1}{SNR_{MFB}}}{\gamma_0 G^*(D^{-*})}$ is a purely anti-causal filter. Therefore, in a manner identical to what we did for the MMSE-LE,

$$
r'_k = (1 - v_0)x_k \underbrace{- \sum_{n \neq 0} v_n x_{k-n} + z'_k}_{e'_k}
$$

Hence,

$$
r'_k = (1 - v_0)x_k + e'_k
$$

Hence, we again have a biased receiver, and the SNR measure that reflects error probability needs to take this into account. Note that since

$$
E(D) = X(D) - R'(D)
$$

In time domain,

$$
e_k = v_0 x_k - e'_k
$$

And therefore for the same reason as in the MMSE-LE, the $SNR_{MMSE-DFE}$ is not the right measure to use for detecting SNR. Hence we need to find $\mathbb{E}[| e'_k |^2]$ and to do that first we find $\mathbb{E}[| e_k |^2]$ as,

$$
\begin{aligned}
\mathbb{E}\left[|e_k|^2\right] &= \sigma^2_{MMSE-DFE} = v_0^2 \mathcal{E}_x + \mathbb{E}\left[|e'_k|^2\right] \\
&= v_0^2 \mathcal{E}_x + \sigma^2_{MMSE-DFE,U}
\end{aligned}
$$

Hence,

$$
\sigma^2_{MMSE-DFE,U} = \sigma^2_{MMSE-DFE} - v_0^2 \mathcal{E}_x = \frac{N_0}{||p||^2 \gamma_0} - v_0^2 \mathcal{E}_x \tag{5.37}
$$

Since from (5.36)

$$
V(D) = \frac{1}{SNR_{MFB}} \frac{1}{\gamma_0} \frac{1}{G^*(D^{-*})}
$$

and $G^*(D^{-*})$ is monic and anti-causal, therefore so is $\frac{1}{G^*(D^{-*})}$,

$$v_0 = \frac{1}{SNR_{MFB}}\frac{1}{\gamma_0} = \frac{N_0}{||p||^2\mathcal{E}_x}\frac{1}{\gamma_0}$$

Therefore

$$\frac{1}{\gamma_0} = \frac{||p||^2\mathcal{E}_x}{N_0}v_0$$

Using this in (5.33) we see that

$$\sigma^2_{MMSE-DFE} = v_0\mathcal{E}_x$$

Inserting this in (5.37), we get

$$\sigma^2_{MMSE-DFE,U} = \mathcal{E}_x v_0 - v_0^2\mathcal{E}_x = v_0\mathcal{E}_x(1-v_0)$$

And hence we get the more informative SNR,

$$SNR_{MMSE-DFE,U} = \frac{(1-v_0)^2\mathcal{E}_x}{v_0(1-v_0)\mathcal{E}_x} = \frac{1}{v_0} - 1 = SNR_{MMSE-DFE} - 1$$

Therefore we have

$$SNR_{MMSE-DFE} = SNR_{MMSE-DFE,U} + 1 \tag{5.38}$$

Notice the similarity in the relationship in (5.38) and (5.27). In both cases the unbiased SNR and the biased SNR differ by 1. This turns out to be a fundamental relationship between the SNR of biased and unbiased detectors.

Note that the error sequence $\{e'_k\}$ is *not* in general white, even with the correct past decisions assumption. Again, the main calculating in finding $SNR_{MMSE-DFE,U}$ is to find the spectral factorization of $\left[Q(D) + \frac{1}{SNR_{MFB}}\right]$ and by using γ_0 found from it we can obtain v_0 and hence $SNR_{MMSE-DFE,U}$.

5.5.2 Zero forcing DFE

We can find the ZF-DFE forward and feedback filters by simply setting $SNR_{MFB} \to \infty$ in the expressions derived in the MMSE-DFE. Therefore we do a spectral factorization of

$$Q(D) = \eta_0 P_c(D) P_c^*(D^{-*})$$

and set the feed forward and feedback filters of the ZF-DFE as,

$$W(D) = \frac{1}{\eta_0 ||p|| P_c^*(D^{-*})}\ ,\ B(D) = P_c(D) \tag{5.39}$$

It is not immediately clear what criterion the zero-forcing DFE is operating on. The zero-forcing DFE is the direct extension of the algebraic argument that led to the ZFE. In the ZFE, we observed that in the absence of noise,

$$Y(D) = ||p||Q(D)X(D) \tag{5.40}$$

and hence we can solve this algebraic equation for x_k by "inverting" the channel, *i.e.,* $W_{ZFE}(D) = \frac{1}{||p||Q(D)}$. The zero-forcing DFE also solves an algebraic equation, where we further assume that previous symbols, $x_{k-1}, \ldots$ are known. The result of solving (5.40) for x_k, given previous symbols is exactly what the ZF-DFE is doing. In order to observe this, let us use (5.39) and examine its effect on r_k and $r_k^{'}$. Since the structure of the ZF-DFE is the same as Figure 5.6, at the output of the feedforward filter we have,

$$\begin{aligned} R(D) &= W(D)Y(D) = \frac{1}{\eta_0 ||p|| P_c^*(D^{-*})} \parallel p \parallel Q(D)X(D) + \frac{1}{\eta_0 ||p|| P_c^*(D^{-*})} Z(D) \\ &\overset{(a)}{=} P_c(D)X(D) + Z^{'}(D) = B(D)X(D) + Z^{'}(D), \end{aligned} \tag{5.41}$$

where (a) follows from (5.39). Now, notice that since $R^{'}(D) = R(D) + [1 - B(D)]\, X(D)$,

$$r_k^{'} = x_k + z_k^{'}, \tag{5.42}$$

which has eliminated the inter-symbol interference. Therefore, in the absence of noise, we would have solved the algebraic equation (5.40) for x_k. Notice that the power spectral density $S_{z^{'}}(D)$ of the modified noise process $Z^{'}(D)$ in (5.42) is,

$$S_{z^{'}}(D) = \frac{1}{\eta_0^2 ||p||^2 P_c(D) P_c^*(D^{-*})} S_z(D) = \frac{N_0 Q(D)}{\eta_0 ||p||^2 Q(D)} = \frac{N_0}{\eta_0 ||p||^2}, \tag{5.43}$$

i.e., it is white! Therefore we have converted the problem into (5.42) which is like the AWGN channel we studied in Chapter 2. However, the caveat is ofcourse that we have assumed perfect decision feedback. We will study how to remove that assumption using a technique described in Section 6.1.

5.6 Fractionally spaced equalization

We have till now assumed perfect synchronization, *i.e.,* we know exactly when the sampling at the output of the WMF occurs. Suppose we think that sampling occurs at kT, but it occurs at $kT + t_0$. Then the equivalent channel is

$$y(kT + t_0) = \sum_m x_m ||p|| q(kT - mT + t_0) + z(kT + t_0)$$

Therefore we are sampling $q(t + t_0)$ at kT, and we design equalizers assuming that it was $q(t)$, hence the equivalent channel is $Q(\omega)e^{-j\omega t_0}$ which could cause loss in performance. One solution is to work with Nyquist sampling and collecting sufficient statistics by this method.

There are several motivations to collect sufficient statistics through Nyquist sampling. One, is as explained above, is due to robustness to timing errors. Another more subtle reason is that in practice, the channel is unknown to the receiver and one needs to estimate the channel. We will learn channel estimation methods in Section 6.3. Therefore, we may not be able to form the matched filter to collect sufficient statistics. Given this, we need a channel independent method to do so, and Nyquist sampling gives us such a tool. By sampling at Nyquist rate, we actually sample at higher than the symbol rate T, and therefore sometimes the terminology "oversampling" is used for this.

Let,

$$y_i(k) = y\left(kT - \frac{iT}{L}\right), i = 0, \ldots, L-1 \tag{5.44}$$

Hence, in transform domain,

$$Y_i(D) = P_i(D)X(D) + Z_i(D), i = 0, \ldots, L-1$$

Where,

$$P_i(D) = \mathcal{D}[p_i(k)]$$

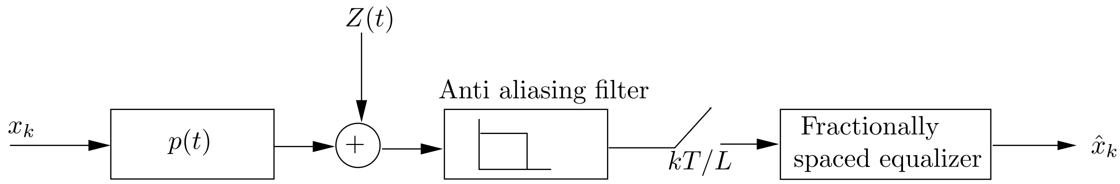


Figure 5.7: Fractionally spaced equalizer.

and

$$p_i(k) = [h(t) * \varphi(t)]_{t=kT-\frac{iT}{L}}$$

and

$$z_i(k) = z\left(kT - \frac{iT}{L}\right)$$

Stacking up all the "oversampled" or fractionally sampled versions, one obtains

$$\mathbf{Y}(D) = \begin{bmatrix} Y_0(D) \\ \vdots \\ Y_{L-1}(D) \end{bmatrix} = \underbrace{\begin{bmatrix} P_0(D) \\ \vdots \\ P_{L-1}(D) \end{bmatrix}}_{\mathbf{P}(D)} X(D) + \underbrace{\begin{bmatrix} Z_0(D) \\ \vdots \\ Z_{L-1}(D) \end{bmatrix}}_{\mathbf{Z}(D)}$$

Which in more compact notation is,

$$\mathbf{Y}(D) = \mathbf{P}(D)X(D) + \mathbf{Z}(D)$$

The equalizer structure can now be a row vector

$$\mathbf{W}(D) = [W_0(D), \ldots, W_{L-1}(D)]$$

and the output of the equalizer is

$$R(D) = \mathbf{W}(D)\mathbf{Y}(D) = \mathbf{W}(D)\mathbf{P}(D)X(D) + \mathbf{W}(D)\mathbf{Z}(D)$$

Again, from basic linear estimation principles, the fractionally spaced MMSE-LE can be found by orthogonality principle,

$$\mathbb{E}\left[E(D)\mathbf{Y}^*(D^{-*})\right] = 0$$

This gives,

$$\mathbb{E}\left[\{X(D) - \mathbf{W}_{MMSE-LE}(D)\mathbf{Y}(D)\}]\mathbf{Y}^*(D^{-*})\right] = 0$$

Yielding,

$$\begin{aligned} \mathbf{W}_{MMSE-LE}(D) &= \mathbb{E}\left[X(D)\mathbf{Y}^*(D^{-*})\right]\mathbb{E}\left[\mathbf{Y}(D)\mathbf{Y}^*(D^{-*})\right]^{-1} \\ &= \mathcal{E}_x\mathbf{P}^*(D^{-*})\left[\mathcal{E}_x\mathbf{P}(D)\mathbf{P}^*(D^{-*}) + LN_0\mathbf{I}\right]^{-1} \end{aligned}$$

Note that the equalization is done and then the output is downsampled, and this alleviates some of the problems of timing offset.

5.6.1 Zero-forcing equalizer

An interesting feature of the fractionally-spaced case is the special case of the zero-forcing equalizer. Here we need

$$\mathbf{W}_{ZF}(D)\mathbf{P}(D) = 1 = \sum_{i=0}^{L-1} W_{i,ZF}(D)P_i(D) \tag{5.45}$$

Now the interesting aspect of this is that one can have $W_i(D)$ which are finite length polynomials and still be able to satisfy (5.45). This implies that for this case one is able to "convert" the channel without infinite length "inverses"! This is really because we only need the downsampled version of the equalizer output to behave like a discrete-time delta function.

Theorem 5.6.1 (Bezout identity). *If $\{P_i(D)\}_{i=0}^{L-1}$ do not share common zeros, then there exists a vector polynomial (of finite degree)*

$$\mathbf{W}(D) = [W_0(D), \ldots, W_{L-1}(D)]$$

such that

$$\mathbf{W}(D)\mathbf{P}(D) = 1$$

Remarks:

1. In fact this condition is a necessary and sufficient condition, *i.e.*, $\{P_i(D)\}_{i=0}^{L-1}$ do not share common zeros (are co-primes) iff there exists a polynomial vector $\mathbf{W}(D)$ such that $\mathbf{W}_{ZF}(D)\mathbf{P}(D) = 1$.

2. This result was shown first by Sylvester (1840) as a test for co-primeness.

3. This shows that there exists a finite-impulse response inverse to the vector channel if the vector channel satisfies the regularity condition of Theorem 5.6.1. Again this inverse is only after "down-sampling".

4. One can also have fractionally spaced decision-feedback equalizer, where the feedforward section operates in fractional spacing and the feedback section is symbol spaced. We will learn more about such structures in the context of finite length equalizers in Section 5.7.

5.7 Finite-length equalizers

In practice one can only implement finite length filters using digital signal processors. One way is to "truncate" the infinite length filters we have derived till now and implement them. However, a more principled approach is to start with the requirement of finite length filtering and derive the optimal finite length filters for equalization. Again, we use oversampled version of the receiver, as done in (5.44).

$$y\left(kT - \frac{iT}{L}\right) = \sum_m x_m p(kT - \frac{iT}{L} - mT) + z\left(kT - \frac{iT}{L}\right), i = 0, \ldots, L-1$$

We define,

$$\mathbf{y}_k = \begin{bmatrix} y(kT) \\ \vdots \\ y\left(kT - \frac{L-1}{L}T\right) \end{bmatrix} = \sum_m x_m \underbrace{\begin{bmatrix} p(kT - mT) \\ \vdots \\ p\left(kT - \frac{L-1}{L}T - mT\right) \end{bmatrix}}_{\mathbf{p}_{k-m}} + \underbrace{\begin{bmatrix} z(kT) \\ \vdots \\ z\left(kT - \frac{L-1}{L}T\right) \end{bmatrix}}_{\mathbf{z}_k} \tag{5.46}$$

Hence in more compact notation,

$$\mathbf{y}_k = \sum_n \mathbf{p}_n x_{k-n} + \mathbf{z}_k$$

Assumption: We impose the condition that the pulse response is of finite duration *i.e.,*

$$p(t) = 0 \qquad \text{for } t \notin [0, \nu T]$$

which implies that

$$\mathbf{p}_k = 0 \qquad \text{for } k < 0 \text{ and } k > \nu.$$

Note that in practice this is a reasonable assumption since most "real-world" channels are *approximately* time-limited.
Thus using this assumption (5.46) becomes

$$\mathbf{y}_k = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_\nu] \begin{bmatrix} x_k \\ \vdots \\ x_{k-\nu} \end{bmatrix} + \mathbf{z}_k.$$

Now, we collect N_f samples of $\mathbf{y}_k$, *i.e.,* a frame of $N_f + \nu$ transmitted symbols as,

$$\mathbf{Y}_k = \begin{bmatrix} \mathbf{y}_k \\ \mathbf{y}_{k-1} \\ \vdots \\ \mathbf{y}_{k-N_f+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \ldots & \mathbf{p}_\nu & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \ldots & \mathbf{p}_\nu & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \ldots & \mathbf{p}_\nu \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-N_f-\nu+1} \end{bmatrix}}_{\mathbf{X}_k} + \underbrace{\begin{bmatrix} \mathbf{z}_k \\ \mathbf{z}_{k-1} \\ \vdots \\ \mathbf{z}_{k-N_f-\nu+1} \end{bmatrix}}_{\mathbf{Z}_k} \tag{5.47}$$

This is the basic model that will be used for finite length equalization. This model is like the basic relationship (5.11) seen in Section 5.2.

5.7.1 FIR MMSE-LE

In this section we derive a finite length MMSE linear equalizer, *i.e.,* the equalizer is restricted to operate on N_f symbol times, or $N_f L$ samples of received sequence. Therefore, the equalizer is a $N_f L$ dimensional row vector applied to the received (sampled) vector $\mathbf{Y}_k$.

$$r_k = \mathbf{w}\mathbf{Y}_k$$

where $\mathbf{w} \in \mathbb{C}^{1\times N_f L}$.
Since one cannot implement a non-causal filter, the system has to be designed for a given delay, *i.e.,* the equalized output r_k, is close to $x_{k-\Delta}$, where Δ is the delay. Note that for infinite filters we did not consider this issue, though a delay was implicitly assumed since we can only implement causal systems. Hence the equalizer works by minimizing the following error:

$$e_k = x_{k-\Delta} - r_k.$$

For the FIR MMSE-LE, the criterion is

$$\mathbf{w}(\Delta) = \arg\min_{\mathbf{w}} \mathbb{E}\left[|e_k|^2\right]$$

Using the orthogonality principle,

$$\mathbb{E}\left[e_k \mathbf{Y}_k^*\right] = 0$$

To find $\mathbf{w}$, we use this orthogonality principle as,

$$\mathbb{E}\left[x_{k-\Delta}\mathbf{Y}_k^*\right] = \mathbf{w}_{opt}(\Delta)\mathbb{E}\left[\mathbf{Y}_k\mathbf{Y}_k^*\right]$$

where

$$\mathbf{R}_{YY} = \mathbb{E}\left[\mathbf{Y}_k \mathbf{Y}_k^*\right]$$

is the auto-correlation matrix.

$$\mathbf{R}_{YX}(\Delta) = \mathbb{E}\left[\mathbf{Y}_k \mathbf{X}_{k-\Delta}^*\right]$$

is the cross-correlation matrix, and hence,

$$\mathbf{w}_{opt}(\Delta) = \mathbf{R}_{XY}(\Delta)\mathbf{R}_{YY}^{-1}$$

In more detail,

$$\begin{aligned}
\mathbf{R}_{XY}(\Delta) &= \mathbb{E}\left[x_{k-\Delta}(\mathbf{X}_k^*\mathbf{P}^* + \mathbf{Z}_k^*)\right] \\
&= [0\ldots0 \quad \underbrace{\mathcal{E}_x}_{(\Delta+1)^{th} position} \quad 0\ldots0]\mathbf{P}^* \\
&= \mathcal{E}_x[0\ldots0 \quad \underbrace{\mathbf{p}_\nu^*}_{(\Delta+1)^{th} position} \quad \ldots\mathbf{p}_0^*, 0\ldots0]
\end{aligned}$$

Let

$$\mathbb{1}_\Delta = [0,\ldots,0,1,0,\ldots,0]^* \in \mathbb{C}^{N_f+\nu},$$

where the 1 occurs in the $(\Delta+1)$th position. Hence,

$$\mathbf{R}_{XY}(\Delta) = \mathcal{E}_x \mathbb{1}_\Delta^* \mathbf{P}^*$$

Also,

$$\begin{aligned}
\mathbf{R}_{YY} &= \mathbb{E}\left[\mathbf{P}\mathbf{X}_k\mathbf{X}_k^*\mathbf{P}^*\right] + \mathbb{E}\left[\mathbf{Z}_k\mathbf{Z}_k^*\right] \\
&= \mathcal{E}_x\mathbf{P}\mathbf{P}^* + LN_0\mathbf{I}_{N_fL}.
\end{aligned}$$

One finds Δ by choosing the setting that minimized $\sigma^2_{FIR-MMSE-LE}$, the MMSE for the FIR equalizer. Now, to compute the MMSE of the finite length equalizer we can write

$$\begin{aligned}
\sigma^2_{MMSE-LE} &= \mathbb{E}[\| x_{k-\Delta} - \mathbf{w}(\Delta)\mathbf{Y}_k \|^2] \\
&= \mathcal{E}_x - \mathbf{R}_{XY}(\Delta)\mathbf{R}_{YY}^{-1}\mathbf{R}_{XY}^*(\Delta) \\
&= \mathcal{E}_x - \mathbf{w}_{opt}(\Delta)\mathbf{R}_{XY}^*(\Delta) \\
&= \mathcal{E}_x - \mathcal{E}_x\mathbb{1}_\Delta^*\mathbf{P}^*\left[\mathcal{E}_x\mathbf{P}\mathbf{P}^* + LN_0\mathbf{I}_{N_fL}\right]^{-1}\mathbf{P}\mathbb{1}_\Delta\mathcal{E}_x \\
&\overset{(a)}{=} \mathbb{1}_\Delta^*\left\{\mathcal{E}_x\mathbf{I}_{N_f+\nu} - \mathcal{E}_x\mathbf{P}^*\left[\mathcal{E}_x\mathbf{P}\mathbf{P}^* + LN_0\mathbf{I}_{N_fL}\right]^{-1}\mathbf{P}\mathcal{E}_x\right\}\mathbb{1}_\Delta
\end{aligned} \tag{5.48}$$

where (a) follows because

$$\mathbb{1}_\Delta^*\mathbb{1}_\Delta = 1.$$

Now the matrix inversion lemma (Woodbury's identity) gives

Lemma 5.7.1. ***(Matrix inversion lemma)*** *Given matrices* $\mathbf{A}, \mathbf{B}, \mathbf{C}$*,*

$$[\mathbf{A} + \mathbf{B}\mathbf{C}\mathbf{B}^*]^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}[\mathbf{C}^{-1} + \mathbf{B}^*\mathbf{A}^{-1}\mathbf{B}]^{-1}\mathbf{B}^*\mathbf{A}^{-1}$$

By identifying,

$$\mathbf{A}^{-1} = \mathcal{E}_x \mathbf{I}_{N_f+\nu}, \;\; \mathbf{B} = \mathbf{P}^*, \;\; \mathbf{C}^{-1} = LN_0 \mathbf{I}_{N_f L}$$

in Lemma 5.7.1 and using this in (5.48), we get,

$$\begin{aligned} \sigma^2_{MMSE-LE} &= \mathbb{1}^*_\Delta \left\{ \mathcal{E}_x^{-1} \mathbf{I}_{N_f+\nu} + \mathbf{P}^*\mathbf{P}\frac{1}{N_0 L} \right\}^{-1} \mathbb{1}_\Delta \\ &= N_0 L \mathbb{1}^*_\Delta \underbrace{\left\{ \mathbf{P}^*\mathbf{P} + \frac{N_0 L}{\mathcal{E}_x} \mathbf{I}_{N_f+\nu} \right\}^{-1}}_{\mathbf{Q}(\Delta)} \mathbb{1}_\Delta \\ &= N_0 L \mathbb{1}^*_\Delta \mathbf{Q}(\Delta) \mathbb{1}_\Delta \end{aligned} \tag{5.49}$$

Therefore the smallest $\sigma^2_{MMSE-LE}$ occurs by choosing the Δ corresponding to the smallest diagonal element of $\mathbf{Q}(\Delta)$. Therefore the optimal delay can be easily identified by just one matrix inversion.

5.7.2 FIR MMSE-DFE

The FIR MMSE-DFE (see Figure 5.8), is similar to the FIR MMSE-LE except that there is now a symbol spaced feedback filter which is also of finite length N_b, which filter the past decisions $\{\hat{x}_{k-\Delta-1}, \ldots, \hat{x}_{k-\Delta-N_b}\}$,

$$e_k = x_{k-\Delta} - \left\{ [b_1, \ldots, b_{N_b}] \begin{bmatrix} \hat{x}_{k-\Delta-1} \\ \vdots \\ \hat{x}_{k-\Delta-N_b} \end{bmatrix} + \mathbf{w}\mathbf{Y}_k \right\}$$

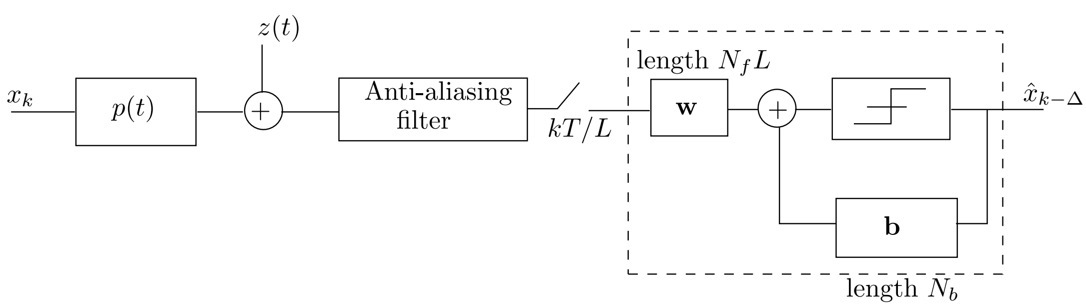


Figure 5.8: Structure of the FIR MMSE-DFE.

As in the earlier MMSE-DFE setting, we derive the DFE assuming correct past decisions.
Let

$$\mathbf{b} = [1, -b_1, -b_2, \ldots, -b_{N_b}]$$

and

$$\mathbf{X}^{k-\Delta}_{k-\Delta-N_b} = \begin{bmatrix} x_{k-\Delta} \\ x_{k-\Delta-1} \\ \vdots \\ x_{k-\Delta-N_b} \end{bmatrix}$$

Therefore we can write,

$$
\begin{aligned}
e_k &= x_{k-\Delta} - \left\{ [b_1, \ldots, b_{N_b}] \begin{bmatrix} \hat{x}_{k-\Delta-1} \\ \vdots \\ \hat{x}_{k-\Delta-N_b} \end{bmatrix} + \mathbf{w}\mathbf{Y}_k \right\} \\
&= [1, -b_1, \ldots, -b_{N_b}] \begin{bmatrix} x_{k-\Delta} \\ \hat{x}_{k-\Delta-1} \\ \vdots \\ \hat{x}_{k-\Delta-N_b} \end{bmatrix} - \mathbf{w}\mathbf{Y}_k \\
&\overset{(a)}{=} \mathbf{b}\mathbf{X}_{k-\Delta-N_b}^{k-\Delta} - \mathbf{w}\mathbf{Y}_k
\end{aligned}
$$

where (a) follows due to perfect decision feedback assumption.
FIR MMSE-DFE criterion:

$$
\begin{aligned}
\{\mathbf{b}_{opt}, \mathbf{w}_{opt}\} &= \arg_{\mathbf{b},\mathbf{w}} \min \mathbb{E}\left[|e_k|^2\right] \\
&= \arg\left\{ \min_{\mathbf{b}} \min_{\mathbf{w}} \mathbb{E}\left[|e_k|^2\right] \right\}
\end{aligned}
$$

That is, we do the minimization in a nested manner just like in the infinite length case. First let us fix $\mathbf{b}$ and find $\mathbf{w}$ in terms of $\mathbf{b}$. To do that, apply orthogonality principle

$$\mathbb{E}\left[e_k \mathbf{Y}^*{}_k\right] = 0$$

which gives,

$$\mathbb{E}\left[\left\{\mathbf{b}\mathbf{X}_{k-\Delta-N_b}^{k-\Delta} - \mathbf{w}\mathbf{Y}_k\right\} \mathbf{Y}_k^*\right] = 0$$

or

$$\mathbf{w}\mathbb{E}\left[\mathbf{Y}_k \mathbf{Y}_k^*\right] = \mathbf{b}\mathbb{E}\left[\mathbf{X}_{k-\Delta-N_b}^{k-\Delta} \mathbf{Y}_k^*\right] \tag{5.50}$$

Now, let us find the terms in (5.50)

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{X}_{k-\Delta-N_b}^{k-\Delta} \mathbf{Y}_k^*\right] &= \mathbb{E}\left[\begin{bmatrix} x_{k-\Delta} \\ \vdots \\ x_{k-\Delta-N_b} \end{bmatrix} \left(\mathbf{P} \begin{bmatrix} x_k \\ \vdots \\ x_{k-N_f-\nu+1} \end{bmatrix} + Z_k \right)^* \right] \\
&= \mathbb{E}\left[\underbrace{\begin{bmatrix} x_{k-\Delta} \\ \vdots \\ x_{k-\Delta-N_b} \end{bmatrix}}_{(N_b+1)\times 1} \underbrace{\left[x_k^* \ldots x_{k-N_f-\nu+1}^*\right]}_{1\times(N_f+\nu)} \right] \underbrace{\mathbf{P}^*}_{(N_f+\nu)\times N_f L} \\
&\overset{\Delta}{=} \mathcal{E}_x \mathbf{J}_\Delta \mathbf{P}^*
\end{aligned}
$$

Now, let us write $\mathbf{J}_\Delta$ in more detail. First let us assume that $x_{k-\Delta}$ occurs in the information window, *i.e.*,

$$\Delta \leq N_f + \nu - 1$$

then $\mathbf{J}_\Delta \neq 0$ for i.i.d. symbols $\{x_k\}$. Now the question is whether $x_{k-\Delta-N_b}$ occurs before or after $x_{k-N_f+1-\nu}$

Case I: $\Delta + N_b \leq N_f + \nu - 1$
Then $x_{k-\Delta-N_b}$ occurs in the information window $\{x_k, \ldots, x_{k-N_f-\nu+1}\}$ and this is the "easy" case.

$$\begin{aligned} \mathbf{J}_\Delta &= \frac{1}{\mathcal{E}_x}\mathbb{E}\begin{bmatrix} x_{k-\Delta} \\ \vdots \\ x_{k-\Delta-N_b} \end{bmatrix}\left[x_k^*, \ldots, x_{k-\Delta}^*, \ldots, x_{k-\Delta-N_b}^*, \ldots, x_{k-N_f-\nu+1}^*\right] \\ &= \begin{bmatrix} \mathbf{0}_{(N_b+1)\times\Delta}, & \mathbf{I}_{(N_b+1)\times(N_b+1)}, & \mathbf{0}_{(N_b+1)\times(N_f+\nu-1-\Delta-N_b)} \end{bmatrix} \end{aligned} \tag{5.51}$$

Case II: $\Delta + N_b > N_f + \nu - 1$
Then $x_{k-\Delta-N_b}$ does not appear in the observation window of information and therefore we have a truncation at the end. Hence the observation looks like,

$$\underbrace{\{x_k, \ldots, x_{k-\Delta}, \ldots, x_{k-N_f-\nu+1}}_{\text{observation window } \mathbf{x}_k}, \ldots, x_{k-\Delta-N_b}\}$$

And we have

$$\mathbf{X}_{k-\Delta-N_b}^{k-\Delta} = \begin{bmatrix} x_{k-\Delta} \\ \vdots \\ x_{k-N_f-\nu+1} \\ \vdots \\ x_{k-\Delta-N_b} \end{bmatrix}$$

Therefore, the $\mathbf{b}$ that needs to be used is also truncated since, $\{x_{k-N_f-\nu}, \ldots, x_{k-\Delta-N_b}\}$ are not part of the observation window. This is really the edge effect that we need to be aware of. One can easily calculate the correct $\mathbf{b}$ for this case by now choosing a shorter decision feedback window size, *i.e.,*

$$\begin{bmatrix} x_{k-\Delta} \\ \vdots \\ x_{k-N_f-\nu+1} \end{bmatrix}$$

of length $(N_f + \nu - \Delta)$, and setting the rest of the taps to zero. For completeness

$$\begin{aligned} \mathbf{J}_\Delta &= \frac{1}{\mathcal{E}_x}\mathbb{E}\left[\begin{bmatrix} x_{k-\Delta} \\ \vdots \\ x_{k-N_f-\nu+1} \\ \vdots \\ x_{k-\Delta-N_b} \end{bmatrix}[x_k^*, \ldots, x_{k-\Delta}^*, \ldots, x_{k-N_f-\nu+1}^*]\right] \\ &= \begin{bmatrix} \mathbf{0}_{(N_f+\nu-\Delta)\times\Delta} & \mathbf{I}_{(N_f+\nu-\Delta)\times(N_f+\nu-\Delta)} \\ \mathbf{0}_{(\Delta+N_b-N_f-\nu+1)\times\Delta} & \mathbf{0}_{(\Delta+N_b-N_f-\nu+1)\times(N_f+\nu-\Delta)} \end{bmatrix} \end{aligned}$$

For $\tilde{\mathbf{b}} = [1, -\tilde{b}_1, \ldots, -\tilde{b}_{N_b}]$,

$$\tilde{\mathbf{b}}\mathbf{J}_\Delta\mathbf{J}_\Delta^*\tilde{\mathbf{b}}^* = \mathcal{E}_x^2\tilde{\mathbf{b}}\tilde{\mathbf{b}}^*$$

Note: The Case II is really an edge effect that we will not consider in detail, but we enunciated it more for completeness. We will focus our attention on Case I.
Now, expressing $\mathbf{w}$ in term of $\mathbf{b}$ we obtain, for Case I,

$$\begin{aligned} \sigma_e^2(\Delta) &= \mathbb{E}\left[\left|\mathbf{b}\mathbf{X}_{k-\Delta-N_b}^{k-\Delta} - \mathcal{E}_x\mathbf{b}\mathbf{J}_\Delta\mathbf{P}^*\mathbf{R}_{YY}^{-1}\mathbf{Y}_k\right|^2\right] \\ &= \mathbf{b}\left\{\mathcal{E}_x I_{(N_b+1)} - \mathcal{E}_x\mathbf{J}_\Delta\mathbf{P}^*\mathbf{R}_{YY}^{-1}\mathbf{P}\mathbf{J}_\Delta^*\mathcal{E}_x\right\}\mathbf{b}^* \end{aligned} \tag{5.52}$$

where,

$$\mathbf{R}_{YY} = \left[\mathcal{E}_x \mathbf{P}\mathbf{P}^* + LN_0 \mathbf{I}_{N_f L}\right]$$

We now use this in (5.52),

$$\sigma_e^2(\Delta) = \mathcal{E}_x \mathbf{b} \left\{ \mathbf{I}_{N_b+1} - \mathbf{J}_\Delta \mathbf{P}^* \left[\mathbf{P}\mathbf{P}^* + \frac{LN_0}{\mathcal{E}_x}\mathbf{I}_{N_f L}\right]^{-1} \mathbf{P}\mathbf{J}_\Delta^* \right\} \mathbf{b}^* \tag{5.53}$$

Now, for case I using (5.51) we have,

$$\mathbf{J}_\Delta \mathbf{J}_\Delta^* = \mathbf{I}_{N_b+1}.$$

Using this in (5.53) we get,

$$\sigma_e^2(\Delta) = \mathcal{E}_x \mathbf{b}\mathbf{J}_\Delta \left\{ \mathbf{I}_{N_f+\nu} - \mathbf{P}^* \left[\mathbf{P}\mathbf{P}^* + \frac{LN_0}{\mathcal{E}_x}\mathbf{I}_{N_f L}\right]^{-1} \mathbf{P} \right\} \mathbf{J}_\Delta^* \mathbf{b}^* \tag{5.54}$$

Now we use the matrix inversion lemma (Lemma 5.7.1) in (5.54) after identifying,

$$\mathbf{A} = \mathbf{I}_{N_f+\nu},\ \mathbf{B} = \mathbf{P}^*,\ \mathbf{C}^{-1} = \frac{LN_0}{\mathcal{E}_x}\mathbf{I}_{N_f L}.$$

Hence using this in (5.54) we get

$$\begin{aligned}
\sigma_e^2(\Delta) &= \mathcal{E}_x \mathbf{b}\mathbf{J}_\Delta \left\{ \mathbf{I}_{N_f+\nu} - \mathbf{P}^* \left[\mathbf{P}\mathbf{P}^* + \frac{LN_0}{\mathcal{E}_x}\mathbf{I}_{N_f L}\right]^{-1} \mathbf{P} \right\} \mathbf{J}_\Delta^* \mathbf{b}^* && (5.55)\\
&= \mathcal{E}_x \mathbf{b} \left[\mathbf{J}_\Delta \left\{ \mathbf{I}_{N_f+\nu} + \frac{\mathcal{E}_x}{LN_0}\mathbf{P}^*\mathbf{P} \right\}^{-1} \mathbf{J}_\Delta^* \right] \mathbf{b}^* \\
&= LN_0 \mathbf{b} \underbrace{\left[\mathbf{J}_\Delta \left\{ \frac{L}{SNR}\mathbf{I} + \mathbf{P}^*\mathbf{P} \right\}^{-1} \mathbf{J}_\Delta^* \right]}_{\tilde{\mathbf{Q}}^{-1}(\Delta)} \mathbf{b}^* \\
&\triangleq LN_0 \mathbf{b} \tilde{\mathbf{Q}}^{-1}(\Delta) \mathbf{b}^* && (5.56)
\end{aligned}$$

Let us define the Choleski decomposition,

$$\begin{aligned}
\tilde{\mathbf{Q}}(\Delta) &= \mathbf{G}_\Delta^* \mathbf{S}_\Delta^{-1} \mathbf{G}_\Delta \\
\tilde{\mathbf{Q}}^{-1}(\Delta) &= \mathbf{G}_\Delta^{-1} \mathbf{S}_\Delta \mathbf{G}_\Delta^{-*}
\end{aligned}$$

where $\mathbf{G}_\Delta$ is an upper triangular matrix, and hence $\mathbf{G}_\Delta^*$ is a lower triangular matrix.

Fact 5.7.1. *If $\mathbf{G}_\Delta$ is an upper triangular matrix, so is $\mathbf{G}_\Delta^{-1}$.*

Hence, the same fact follows for lower triangular matrices. Therefore, $\mathbf{G}_\Delta^{-1}$ is upper triangular and $\mathbf{G}_\Delta^{-*}$ is lower triangular. Now, suppose we have

$$\mathbf{S}_\Delta = \begin{bmatrix} s_0(\Delta) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_{N_b}(\Delta) \end{bmatrix} \tag{5.57}$$

as the diagonal matrix $\mathbf{S}_\Delta$, with the ordering property,

$$s_0(\Delta) \leq s_1(\Delta) \leq \ldots \leq s_{N_b}(\Delta)$$

This property is assured due to the Choleski factorization. Hence,

$$\begin{aligned} \sigma_e^2(\Delta) &= LN_0\mathbf{b}\left[\mathbf{G}_\Delta^{-1}\mathbf{S}_\Delta\mathbf{G}_\Delta^{-*}\right]\mathbf{b} \\ &= LN_0(\mathbf{b}\mathbf{G}_\Delta^{-1})\mathbf{S}_\Delta(\mathbf{b}\mathbf{G}_\Delta^{-1})^* \end{aligned} \tag{5.58}$$

To minimize this we want to pick off $s_0(\Delta)$, *i.e.,* the top-left corner element of $\mathbf{S}_\Delta$, for which we want

$$\mathbf{b}\mathbf{G}_\Delta^{-1} = [1, 0, \ldots, 0]$$

This means that we want $\mathbf{b}$ to be the first row of $\mathbf{G}_\Delta$. This can be seen by noticing that,

$$\mathbf{G}_\Delta\mathbf{G}_\Delta^{-1} = \mathbf{I}$$

If

$$\mathbf{G}_\Delta = \begin{bmatrix} \mathbf{g}_\Delta(0) \\ \vdots \\ \mathbf{g}_\Delta(N_b) \end{bmatrix},$$

then clearly

$$\mathbf{g}_\Delta(0)\mathbf{G}_\Delta^{-1} = [1, 0, \ldots, 0]$$

If $\mathbf{g}_\Delta(0)$ is the top row of upper triangular matrix $\mathbf{G}_\Delta$

$$\mathbf{b}_{opt} = \mathbf{g}_\Delta(0) \tag{5.59}$$

and

$$\begin{aligned} \mathbf{w}_{opt} &= \mathbf{b}_{opt}\mathbf{R}_{XY}(\Delta)\mathbf{R}_{YY}^{-1} \\ &\overset{(b)}{=} \underbrace{\mathbf{g}_\Delta(0)\mathbf{J}_\Delta(\mathbf{P}^*\mathbf{P} + \frac{L}{SNR}\mathbf{I})^{-1}}_{feed\,forward\,filter} \underbrace{\mathbf{P}^*}_{matched\,filter} \end{aligned}$$

where (b) follows because,

$$\begin{aligned} \mathbf{P}^*\mathbf{P}\mathbf{P}^* + \mathbf{P}^*\frac{L}{SNR} &= \mathbf{P}^*\mathbf{P}\mathbf{P}^* + \frac{L}{SNR}\mathbf{P}^* \\ \mathbf{P}^*\left(\mathbf{P}\mathbf{P}^* + \frac{L}{SNR}\mathbf{I}\right) &= \left(\mathbf{P}^*\mathbf{P} + \frac{L}{SNR}\mathbf{I}\right)\mathbf{P}^* \\ \Rightarrow \left(\mathbf{P}^*\mathbf{P} + \frac{L}{SNR}\mathbf{I}\right)^{-1}\mathbf{P}^* &= \mathbf{P}^*\left(\mathbf{P}\mathbf{P}^* + \frac{L}{SNR}\mathbf{I}\right)^{-1} \end{aligned}$$

Hence, we basically have the same interpretation of the FIR MMSE-DFE as in the infinite length case as a matched filter cascaded with a feed forward filter.
Now, for the performance of the FIR MMSE-DFE, from (5.58) and (5.59), we see that

$$\sigma_e^2(\Delta) = LN_0 s_0(\Delta)$$

where $s_0(\Delta)$ is defined in (5.57). Hence, both the settings of the FIR MMSE-DFE and the analysis involves the Choleski decomposition of $\tilde{\mathbf{Q}}(\Delta)$ defined in (5.56). This is similar to the analysis of the FIR MMSE-LE. However, the optimization with respect to the decision delay, Δ, needs to be carried out separately unlike the FIR MMSE-LE. Therefore for different values of Δ, the $\sigma_e^2(\Delta)$ is evaluated and the delay corresponding to the minimum is chosen.

5.8 Problems

Problem 5.1

Assume that $\mathbf{Y}(k), \mathbf{X}(\mathbf{k}), \mathbf{Z}(\mathbf{k}) \in \mathbb{R}^2$. Given the model $\mathbf{Y}(k) = \mathbf{H}\mathbf{X}(k) + \mathbf{Z}(k)$, where $\mathbf{X}(k), \mathbf{Z}(k)$ are independent zero mean Gaussian random vectors with covariance matrices $\mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{I}$ respectively. Let $\mathbf{H} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Find the optimal (MMSE criterion) $\mathbf{W}$ such that $\hat{\mathbf{X}}(k) = \mathbf{W}\mathbf{Y}(k)$. Find σ^2_{MMSE}.

Problem 5.2

Assume that $\mathbf{Y}(k) \in \mathbb{C}^N$ be a noisy observation of $\mathbf{X}(k) \in \mathbb{C}^N$. We allow dependency between $\mathbf{Y}(k), \mathbf{X}(l)$ where $k \neq l$. We observe the complete sequence $\{\mathbf{Y}(k)\}$ and would like to estimate $\{\mathbf{X}(k)\}$ by passing it through a filter $\mathbf{W}(k) \in \mathbb{C}^{N \times N}$ i.e.

$$\hat{\mathbf{X}}(k) = \sum_n \mathbf{W}(n)\mathbf{Y}(k-n).$$

Let $\mathbf{S}_{\mathbf{XY}}(D) = \mathbb{E}[\mathbf{X}(D)\mathbf{Y}^*(\frac{1}{D^*})]$, $\mathbf{S}_{\mathbf{XY}}(D), \mathbf{S}_{\mathbf{YY}}(D) \in \mathbb{C}^{N\times N}$. Find the optimum filter $\mathbf{W}(D)$ (in the MMSE sense) in terms of $\mathbf{S}_{\mathbf{XY}}(D)$ and $\mathbf{S}_{\mathbf{YY}}(D)$.
Note that $\hat{\mathbf{X}}(k)$ is obtained by using the complete noisy sequence $\{\mathbf{Y}(k)\}_{k=-\infty}^{\infty}$. This operation is called as smoothing.
Hint: For the D-transform of vectors or matrices, we take D-transform of individual components.

Problem 5.3

1. Let the autocorrelation function of $x(k)$ is given by $\phi_x(l) = e^{-2|l|}$. Find the optimal (MMSE) linear prediction filter for $x(k)$, i.e.

$$\hat{x}(k) = \sum_{i=1}^{\infty} a_i x(k-i).$$

2. We know that in the prediction problem the estimate $\hat{x}(k) = \sum_{i=1}^{\infty} a_i x_{k-i}$ depends on the past. In this problem which is an instance of a filtering problem, we would like to estimate $\{x(k)\}$ based on the noisy observations $\{y(k)\}$ of the past and present by filtering i.e.

$$\hat{x}(k) = \sum_{i=0}^{\infty} a_i y_{k-i}.$$

   Let the autocorrelation function of $x(k)$ be $\phi_x(l) = e^{-2|l|}$. Also

$$y(k) = x(k) + z(k),$$

   where $z(k)$ is zero-mean unit-variance white Gaussian noise. Find the optimum $\{a_i\}_{i=0}^{\infty}$ (in the MMSE sense).

   Note: In the filtering problem we only use the noisy observation of the past and present, while in the smoothing problem (problem 2) we also use the future noisy observations.

   Hint: Use orthogonality principle.

Problem 5.4

Suppose we are given $S_x(z) = 1 - b^\star D^{-1} - bD + |b|^2$. Suppose the Paley-Wiener condition holds. Write the spectral factorization of $S_x(D)$. Depending on b, find the the linear filter used to do a linear prediction from the sequence $\{x_k\}$.

Problem 5.5

Given some realizations $\{y_n\}$, we want to estimate $\hat{x}$ by using a linear filter. Derive the orthogonality principle $E[e_{opt}(k)y^\star(k)] = 0$, and determine the coefficients of the filter $h_{k,n}$ from the MMSE estimation problem.

Problem 5.6

Given the following model $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k) + \mathbf{z}(k)$ where $\mathbf{z}(k)$ is a white Gaussian noise vector. Find the optimal $\mathbf{W}$ such that $\hat{x}(k) = \mathbf{W}\mathbf{y}(k)$.

Problem 5.7

Given two WSS random sequences $\{x_k\}$ and $\{y_k\}$, we want to use $\{y_l\}_{l=-\infty}^{k}$ to predict $\{x_k\}$ by using a linear filter, i.e $\hat{x}_k = \sum_{l=0}^{\infty} a_l y_{k-l}$. Find $\{a_l\}$.

Problem 5.8

[Linear Estimation] Consider the additive noise model given in Fig. 5.9. Let X, Y_1, Y_2, Z_1, $Z_2 \in \mathbb{C}$,

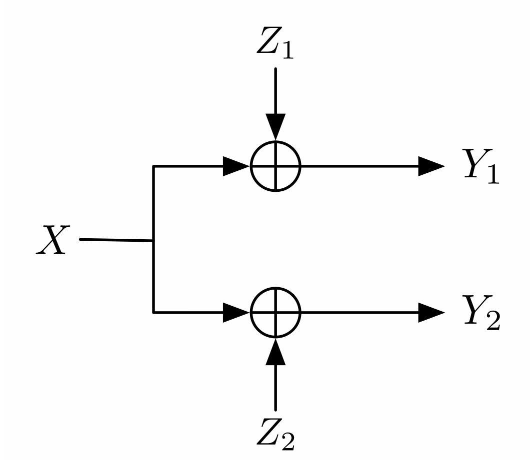


Figure 5.9: Additive Noise Channel

i.e complex random variables. Moreover, let $\mathbb{E}X = 0 = \mathbb{E}Z_1 = \mathbb{E}Z_2$, and Z_1 and Z_2 are independent of X.

1. Given Y_1, Y_2 find the best minimum mean squared error *linear* estimator $\widehat{X}$, where the optimization criterion is $\mathbb{E}|X - \widehat{X}|^2$.

   Assume the following: $\mathbb{E}|X|^2 = \mathcal{E}_x$, $\mathbb{E}|Z_1|^2 = \mathbb{E}|Z_2|^2 = 1$, $\mathbb{E}Z_1 Z_2^* = \frac{1}{\sqrt{2}}$

2. If $\mathbb{E}Z_1 Z_2^* = 1 = \mathbb{E}Z_2 Z_1^*$, what is the best MMSE linear estimator of X ?

Problem 5.9

[Combining linear estimators] Let Y_a and Y_b be two separate observations of a zero mean random variable X such that

$$\begin{aligned} Y_a &= H_a X + V_a \\ \text{and} \quad Y_b &= H_b X + V_b, \end{aligned}$$

where $\{V_a, V_b, X\}$ are mutually independent and zero-mean random variables, and $V_a, V_b, X, Y_a, Y_b \in \mathbb{C}$.

1. Let $\widehat{X}_a$ and $\widehat{X}_b$ denote the linear MMSE estimators for X given Y_a and Y_b respectively. That is

$$\begin{aligned} W_a &= \text{argmin}_{W_a} \mathbb{E}\left[||X - W_a Y_a||^2\right], \\ W_b &= \text{argmin}_{W_b} \mathbb{E}\left[||X - W_b Y_b||^2\right] \end{aligned}$$

and

$$\widehat{X}_a = W_a Y_a \quad \text{and} \quad \widehat{X}_b = W_b Y_b.$$

Find $\widehat{X}_a$ and $\widehat{X}_b$ given that

$$\mathbb{E}\left[XX^*\right] = \sigma_x^2, \mathbb{E}\left[V_a V_a^*\right] = \sigma_a^2, \mathbb{E}\left[V_b V_b^*\right] = \sigma_b^2.$$

Also, find the error variances,

$$\begin{aligned} P_a &= \mathbb{E}\left[(X - \widehat{X}_a)(X - \widehat{X}_a)^*\right] \\ P_b &= \mathbb{E}\left[(X - \widehat{X}_b)(X - \widehat{X}_b)^*\right] \end{aligned}$$

2. We, have the following identities,

$$\mathbf{r}_x \mathbf{h}^* \left[\mathbf{h}\mathbf{r}_x\mathbf{h}^* + \mathbf{r}_v\right]^{-1} = \left[\mathbf{r}_x^{-1} + \mathbf{h}^*\mathbf{r}_v^{-1}\mathbf{h}\right]^{-1} \mathbf{h}^*\mathbf{r}_v^{-1} \tag{5.60}$$

$$\mathbf{r}_x - \mathbf{r}_x\mathbf{h}^* \left[\mathbf{h}\mathbf{r}_x\mathbf{h}^* + \mathbf{r}_v\right]^{-1} \mathbf{h}\mathbf{r}_x = \left[\mathbf{r}_x^{-1} + \mathbf{h}^*\mathbf{r}_v^{-1}\mathbf{h}\right]^{-1} \tag{5.61}$$

Prove that

$$P_a^{-1}\widehat{X}_a = \frac{H_a^*}{\sigma_a^2} Y_a, \quad P_b^{-1}\widehat{X}_b = \frac{H_b^*}{\sigma_b^2} Y_b. \tag{5.62}$$

and

$$P_a^{-1} = \frac{1}{\sigma_x^2} + \frac{H_a H_a^*}{\sigma_a^2}, \quad P_b^{-1} = \frac{1}{\sigma_x^2} + \frac{H_b H_b^*}{\sigma_b^2}. \tag{5.63}$$

Hint: Use identities (5.60), (5.61).

3. Now we find the estimator $\widehat{X}$, given both observations Y_a and Y_b, i.e.,

$$\begin{pmatrix} Y_a \\ Y_b \end{pmatrix} = \begin{pmatrix} H_a \\ H_b \end{pmatrix} X + \begin{pmatrix} V_a \\ V_b \end{pmatrix}.$$

We want to find the linear MMSE estimate

$$\widehat{X} = \begin{pmatrix} U_a & U_b \end{pmatrix} \begin{pmatrix} Y_a \\ Y_b \end{pmatrix},$$

where

$$\begin{pmatrix} U_a & U_b \end{pmatrix} = \text{argmin}_{(U_a,U_b)} \mathbb{E}\left[||X - \widehat{X}||^2\right]$$

and define the corresponding error variance

$$P = \mathbb{E}\left[(X - \widehat{X})(X - \widehat{X})^*\right].$$

Use (5.62), (5.63) to show that

$$\begin{aligned} P^{-1}\widehat{X} &= P_a^{-1}\widehat{X}_a + P_b^{-1}\widehat{X}_b \\ \text{and} \quad P^{-1} &= P_a^{-1} + P_b^{-1} - \frac{1}{\sigma_x^2}. \end{aligned}$$

Problem 5.10

[Noise cancellation] Consider two individual scalar observations Y_1, Y_2 as,

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= Z_2, \end{aligned} \tag{5.64}$$

Assume that zero mean Z_1, Z_2 are independent of X and are correlated with covariance,

$$\mathbb{E}[\mathbf{Z}\mathbf{Z}^*] = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho^*\sigma^2 & \sigma^2 \end{bmatrix}, \tag{5.65}$$

where $\mathbf{Z} = [Z_1, Z_2]^T$. Let $\mathbb{E}[|X|^2] = \mathcal{E}_x$.

(a) Find the best linear MMSE estimate of $\hat{X}$ of the random variable X from the observations Y_1, Y_2.

(b) What is the minimum mean-squared error $\mathbb{E}[|X - \hat{X}|^2]$ of the best linear estimator? Is there a value of ρ for which we get $\mathbb{E}[|X - \hat{X}|^2] = 0$? Interpret the result if $\mathbb{E}[|X - \hat{X}|^2] = 0$ is possible.

(c) Find the best estimate $\hat{Z}_1$ of the random variable Z_1 from Y_2. Consider the operation

$$\tilde{Y}_1 = Y_1 - \hat{Z}_1$$

Find the best linear estimate of X from $\tilde{Y}_1$. Is it the same as the answer you found in (a)? Do you have an interpretation?

Problem 5.11

Assume that the channel model is

$$Y_k = X_k + X_{k-1} + Z_k$$

where $X_k \in \{-1, 1\}$ and Z_k is i.i.d Gaussian noise of unit variance. Find the minimum distance between two possible paths on the MLSE trellis.

Problem 5.12

Consider the discrete time channel model

$$y_n = \sum_k \mathcal{R}_g(k) x_{n-k} + z_n,$$

where z_n is a complex valued circularly symmetric Gaussian process with $\mathcal{R}_z(k) = N_0\mathcal{R}_g(k)$. Assume we filter this received signal through some filter $F(z)$.

1. How do we have to choose F in order to eliminate the intersymbol interference completely.
2. What is the power spectral density of the noise at the output of the filter.

Problem 5.13

Consider the channel $Y(D) = ||p||Q(D)X(D) + Z(D)$ with

$$\begin{aligned} ||p||^2 &= 1 + aa^* \\ Q(D) &= \frac{a^*D^{-1} + ||p||^2 + aD}{||p||^2} \\ 0 \le |a| &< 1. \end{aligned}$$

- Find the zero forcing and minimum mean square error equalizers $W_{\text{ZFE}}(D)$ and $W_{\text{MMSE-LE}}(D)$. Use the variable $b = ||p||^2 \left(1 + \frac{1}{SNR_{MFB}}\right)$ in your expression for $W_{\text{MMSE-LE}}(D)$.
- Find the roots r_1, r_2 of the polynomial

$$aD^2 + bD + a^*.$$

  Show that $b^2 - 4aa^*$ is always a real positive rumber (for $|a| \neq 1$). *Hint:* Consider the case where $\frac{1}{SNR_{MFB}} = 0$. Let r_2 be the root for which $|r_2| < |r_1|$. Show that $r_1 r_2^* = 1$.
- Use the previous results to show that for the MMSE-LE

$$W(D) = \frac{||p||}{a} \frac{D}{(D - r_1)(D - r_2)} = \frac{||p||}{a(r_1 - r_2)} \left(\frac{r_1}{D - r_1} - \frac{r_2}{D - r_2} \right).$$

- Find $w(0)$.
- Show that the canonical factorization is

$$Q(D) + \frac{1}{SNR_{MFB}} = \gamma_0 (1 - r_2 D^{-1})(1 - r_2^* D).$$

  What is γ_0 in terms of a and b?
- Find $B(D)$ and $W(D)$ for the MMSE-DFE.
- Find expressions for SNR_{ZFE}, $SNR_{MMSE-LE-U}$, $SNR_{MMSE-LE}$, $SNR_{MMSE-DFE-U}$ and $SNR_{MMSE-DFE}$.

Problem 5.14

Consider the following system where $\sigma_n^2 = 0.1$, $\mathcal{E}_x = 1$, $l = 1$:

$$\phi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}(\frac{t}{T})$$
$$h(t) = \delta(t) - 0.5\delta(t - T)$$

1. We assume perfect anti-aliasing filtering wih gain $\sqrt{T}$. Find $p(t)$ corresponding to the discrete time-channel:

$$y_k = x_k - 0.5x_{k-1} + n_k$$

   Also find the matrix P.
2. Find a 3 tap FIR MMSE-LE for $\Delta = 2$.
3. Find the $\sigma^2_{MMSE-LE}$ for the equalizer of the previous part.
4. Design an MMSE-DFE which has 2 feedforward taps and 1 feedback tap. Assume that $\Delta = 1$.
5. Find the $\sigma^2_{MMSE-DFE}$ for the equalizer of the previous part.

Problem 5.15

Consider transmission over an ISI channel with 2-PAM constellation of unit energy and symbol period T. Let $\varphi(t) = \frac{1}{\sqrt{T}}\text{sinc}\left(\frac{t}{T}\right)$ and $h(t) = \delta(t) - \frac{1}{2}\delta\left(t - T\right)$. Assume that AWGN noise has power spectral density $N_0/2 = 0.01$ (this is the problem-2 of homework-4, you can use necessary results). Recall that after matched filtering the model is $Y(D) = ||p||Q(D)X(D) + Z(D)$. Find the ZFE, MMSE-LE and MMSE-DFE equalizers for this model. Find SNR_{ZFE}, $\text{SNR}_{\text{MMSE-LE}}$ and $\text{SNR}_{\text{MMSE-DFE}}$.

Problem 5.16

Consider the channel model discussed in class $Y(D) = ||p||Q(D)X(D) + Z(D)$ with

$$\begin{aligned} \mathcal{E}_x &= 1 \\ N_0 &= 0.01 \\ ||p||^2 &= 1.25 \\ Q(D) &= \frac{2}{5}D^{-1} + 1 + \frac{2}{5}D \end{aligned}$$

1. Find $B(D)$ and $W(D)$ for the ZF-DFE.
2. Find SNR_{ZF-DFE}.

Problem 5.17

Consider the channel $Y(D) = ||p||Q(D)X(D) + Z(D)$ with

$$\begin{aligned} \mathcal{E}_x &= 1 \\ N_0 &= 0.01 \\ ||p||^2 &= 1 + aa^* \\ Q(D) &= \frac{a^*D^{-1} + ||p||^2 + aD}{||p||^2} \end{aligned}$$

$$0 \leq |a| < 1.$$

1. Find η_0 and $P_c(D)$ such that $Q(D) = \eta_0 P_c(D)P_c^*(D^{-*})$
2. Find $B(D)$ and $W(D)$ for the ZF-DFE.
3. Find SNR_{ZF-DFE}.

Problem 5.18

[Equalization] Given a channel with transfer function $H(Z) = 1 + cz^{-1}$ with $|c| < 1$, additive white Gaussian noise of variance σ_z^2 and channel input $\{x_k\}$ with $\mathbb{E}|x_k|^2 = 1$.

1. Find the zero forcing linear equalizer $W_{ZFE}(D)$. Find the expression of SNR_{ZFE}.
2. Find the MMSE linear equalizer $W_{MMSE-LE}(D)$. Find the expression of $SNR_{MMSE-LE}$.
3. Find the MMSE-DFE equalizer, i.e. $B_{MMSE-DFE}(D)$ and $W_{MMSE-DFE}(D)$. Also find an expression for $SNR_{MMSE-DFE}$.

Hint: the integral relations in the Instructions may be of use here.

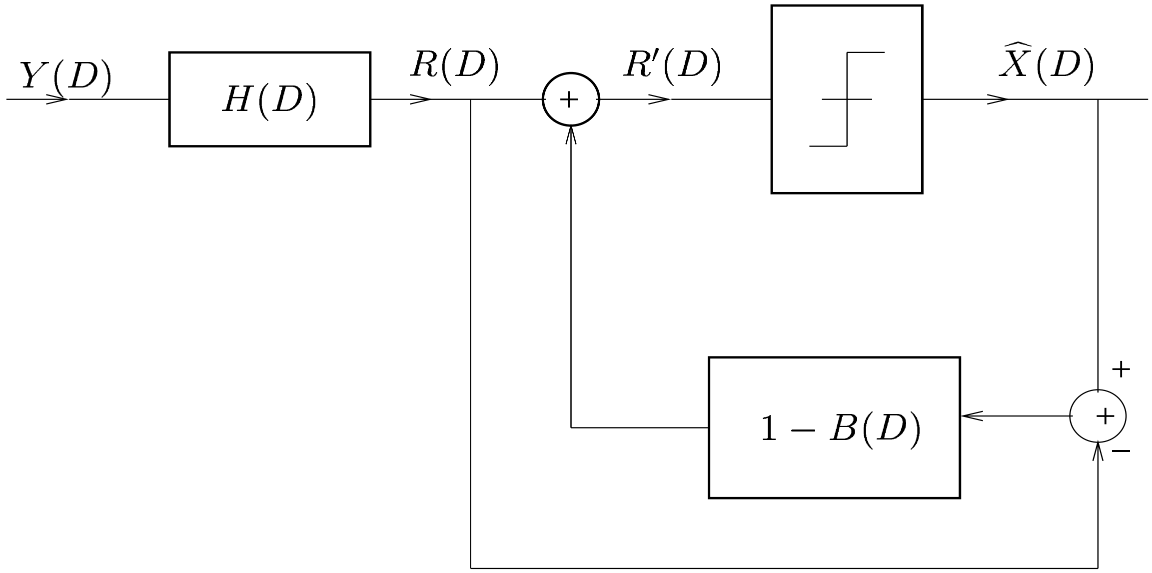


Figure 5.10: Block diagram for noise prediction DFE

Problem 5.19

[Noise prediction DFE] Consider the discrete time model studied in class

$$Y(D) = ||p||Q(D)X(D) + Z(D),$$

where $S_x(D) = \mathcal{E}_x$, $S_z(D) = N_0 Q(D)$ with $Q(D) = Q^*(D^{-*})$. In class we derived the MMSE-DFE, but in this problem we consider a slightly different structure shown in Figure 5.10. As in class we consider perfect decision feedback, i.e., all past decisions are correct. Let

$$\begin{aligned} R(D) &= H(D)Y(D), \\ R'(D) &= R(D) + (1 - B(D))(X(D) - R(D)). \end{aligned}$$

We restrict $B(D)$ to be causal and monic, i.e.,

$$B(D) = 1 + \sum_{l=1}^{\infty} b_l D^l.$$

We choose $H(D)$ and $B(D)$ to minimize

$$\mathbb{E}\left[|x_k - r'_k|^2\right]$$

as we did in class.

1. Find $H(D)$ in terms of $B(D)$ by using orthogonality principle.
2. Set-up the prediction problem by proving that the error

   $$E(D) = X(D) - R'(D) = B(D)X(D) - B(D)H(D)Y(D).$$

   Use the solution of $H(D)$ in terms of $B(D)$ found in part (1) to show that

   $$E(D) = B(D)U(D)$$

and find the expression for $U(D)$.

It can be shown (you don't have to do this) that

$$S_U(D) = \frac{N_0/||p||^2}{Q(D) + 1/SNR_{MFB}}.$$

Given this can you comment on the values of $H(D)$ and $B(D)$ with respect to the quantities derived in class. In particular, is the noise-prediction DFE the same as the MMSE-DFE derived in the class?

3. If $B(D) = 1$, what does the structure in Figure 5.10 become?

Problem 5.20

[*The Oversampled ZF equalizer*] Let

$$\begin{bmatrix} Y_1(D) \\ Y_2(D) \end{bmatrix} = \begin{bmatrix} 1+0.9D \\ 1+0.8D \end{bmatrix} X(D) + \begin{bmatrix} Z_1(D) \\ Z_2(D) \end{bmatrix}$$

i.e. no common zeros among the oversampled responses. Find the ZFE for this case. This should turn out to be a FIR channel since the $P_i(D)$ do not share a common zero.

Problem 5.21

Consider the following system where $\sigma_n^2 = 0.1$, $\mathcal{E}_x = 1$, the oversampling factor $L = 2$:

$$\phi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}(\frac{t}{T})$$

$$h(t) = \delta(t) - 0.5\delta(t-T)$$

1. We assume perfect anti-aliasing filtering wih gain $\sqrt{T}$. Find $p(t)$ at the output of the anti-aliasing filter. Find $p(t)$ corresponding to the discrete time-channel:

$$\mathbf{y}_k = \mathbf{p}_0 x_k + \mathbf{p}_1 x_{k-1} + \mathbf{z}_k$$

   Also find $[\mathbf{p}_0, \mathbf{p}_1]$.

2. We want to design a 4 tap FIR MMSE-LE equalizer. Find the Δ which minimizes $\sigma^2_{MMSE-LE}$ and for that Δ find the corresponding equalizer. Also find $\sigma^2_{MMSE-LE}$.

3. Design an FIR MMSE-DFE which has 4 feedforward taps and 1 feedback tap. Assume that $\Delta = 1$. Also find $\sigma^2_{MMSE-DFE}$.

Problem 5.22

Consider the following signal model:

$$\mathbf{Y_k} = \mathbf{P}\mathbf{Y_k} + \mathbf{Z_k}$$

where $\mathbf{Z_k}, \mathbf{Y_k} \in \mathbb{C}^{\mathbf{N_f L}}$, $\mathbf{P} \in \mathbb{C}^{(\mathbf{N_f L})\times(\mathbf{N_f}+\nu)}$, $\mathbf{X} \in \mathbb{C}^{\mathbf{N_f}+\nu}$

1. Using the orthogonality principle, derive the FIR MMSE-LE equalizer.

2. Find the expression of $\sigma^2_{MMSE-LE}$.

3. How can we choose the value of Δ?

Problem 5.23

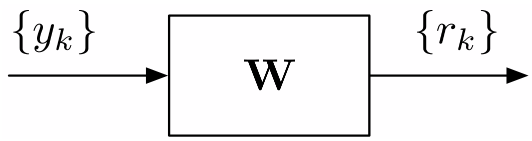
[TARGET CHANNELS] Suppose we have a linear time-invariant channel i.e

$$\mathbf{y}_k = \sum_{n=0}^{\nu} \mathbf{p}_n x_{k-n} + \mathbf{z}_k$$

where $\mathbf{y}_k, \mathbf{p}_n, \mathbf{z}_k \in \mathbb{C}^L$ and $x_k \in \mathbb{C}$. In class we developed the following block model for N_f samples of $\{y_k\}$:

$$\mathcal{Y}_k = \mathbf{P}\mathcal{X}_k + \mathcal{Z}_k$$

where $\mathcal{Y}_k$, $\mathcal{Z}_k \in \mathbb{C}^{N_f L}$, $\mathcal{X}_k \in \mathbb{C}^{N_f}$, $\mathbf{P} \in \mathbb{C}^{N_f L \times (N+\nu)}$. Suppose we want the following receiver structure and we want $r_k \approx \sum_{n=0}^{\eta} h_n x_{k-\Delta-n}$, where $\eta \ll \nu$. That is we want the output of the equalizer to be



close to a given target channel $\{h_n\}$. Therefore we find $\mathbf{W} \in \mathbb{C}^{1 \times N_f L}$:

$$\mathbf{W}_{opt} = \arg\min_{\mathbf{W}} \mathbb{E}||\mathbf{W}\mathcal{Y}_k - \mathbf{H}\mathcal{X}_k||^2 \tag{5.66}$$

where $\mathbf{H} = [\underbrace{0 \cdots 0}_{\Delta \text{ times}} \;\; h_0 \;\; h_1 \;\; \cdots \;\; h_\eta \;\; \underbrace{0 \cdots 0}_{N_f+\nu-\eta-1}]$ i.e $\mathbf{H} \in \mathbb{C}^{1 \times (N_f+\nu)}$ You may assume that $\mathbb{E} x_k x_{k-\ell} = \mathcal{E}_x \delta_\ell$, $\mathbb{E}|z_k|^2 = \sigma^2$ and $\{z_k\}$ is AWGN.

1. (Complete equalization) In class we derived the finite length equalizer when

$$h_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.67}$$

   Re-derive the optimal MMSE finite length equalizer $\mathbf{W}$, for this case, i.e. for $\{h_n\}$ given in (5.67) and the criterion in (5.66) (i.e. same as done in class).

2. Now if we want a particular target channel $\{h_n\}_{n=0}^{\eta}$, and the criterion is given in (5.66), derive the optimal filter $\mathbf{W}_{\text{opt}}$ for the given target channel.

3. For a given target channel $\{h_n\}_{n=0}^{\eta}$, compute the $\sigma^2_{\text{FIR-MMSE-LE}}$, i.e.

$$\mathbb{E}||\mathbf{W}_{\text{opt}}\mathbf{Y}_k - \mathbf{H}\mathcal{X}_k||^2$$

Problem 5.24

[TARGET CHANNELS] Suppose we have a linear time invariant channel done in class, i.e.,

$$Y(D) = ||p||Q(D)X(D) + Z(D),$$

with $Q(D) = Q^*(D^{-*})$. Also there is another process $U(D) = H(D)X(D)$, which we want to estimate.

1. Given observations $\{y_k\}$, find the linear estimator

$$\widehat{U}(D) = W(D)Y(D)$$

   which minimizes the mean-squared error, i.e.,

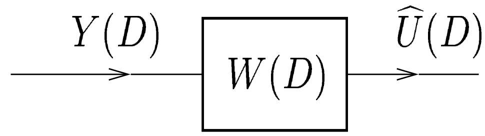


Figure 5.11: MMSE estimation

$$W(D) = \text{argmin}_{W(D)} \mathbb{E}\left[||u_k - \widehat{u}_k||^2\right].$$

You can assume that $\{X_k\}$ and $\{Z_k\}$ are independent and that

$$S_x(D) = \mathcal{E}_x, S_z(D) = N_0 Q(D).$$

2. Given the optimum linear MMSE estimator given in part (1) we define the error as

$$e_k = u_k - \widehat{u}_k.$$

Find the power spectral density of $\{e_k\}$, $S_E(D)$.

3. If $H(D) = 1$, can you comment on the operation performed in part (1)?
Hint: Is it related to any structure studied in class?

Chapter 6

Transmission structures

In Chapter 4 and 5, we learnt detection for ISI channels. However, the transmission structure remained the same as an AWGN channel. This chapter examines techniques at the transmitter that are suitable for ISI channels. We study precoding in Section 6.1 which also mitigates error-propagation effects of the DFE. Section 6.2 demonstrates a canonical transmission and reception structure which is suitable for ISI channels. Finally, Section 6.3 is devoted to training sequence optimization for channel estimation.

6.1 Pre-coding

In order to counter error-propagation in DFE's, one method is to do "precoding". The first idea is to move the feedback section of the DFE to the transmitter. Note that this requires the knowledge of the channel at the transmitter. We will also study another transmit strategy in this chapter which uses alternate way of simplifying the equalization problem (multi carrier transmission).

6.1.1 Tomlinson-Harashima precoding

For simplicity we will first illustrate this with real (*i.e.,* one-dimensional) baseband signals.

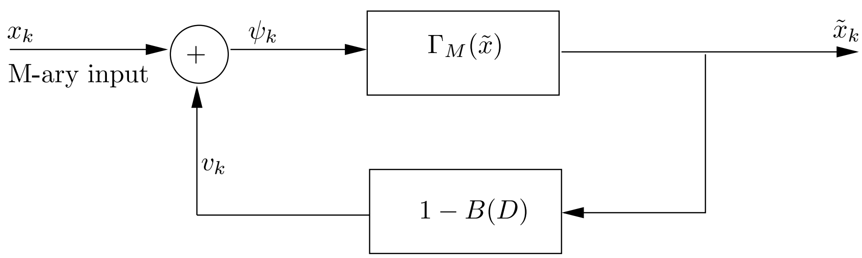


Figure 6.1: The Tomlinson-Harashima precoder for real baseband signals.

The basic idea is understood by examining the ZF-DFE. In the ZF-DFE as seen in Section 5.7.2, we have,

$$W(D) = \frac{1}{\gamma_0 ||p|| P_c^*(D^{-*})} \; , \; B(D) = P_c(D) \tag{6.1}$$

where,

$$Q(D) = \gamma_0 P_c(D) P_c^*(D^{-*}),$$

and $B(D)$ is monic, *i.e.,* $b_0 = 1$. Now, suppose we remove the $\Gamma_M(\cdot)$ block from Figure 6.1, then we get,

$$\psi_k = x_k - \sum_{l=1}^{\infty} b_l \psi_{k-l} \tag{6.2}$$

Therefore we get,

$$\sum_{l=0}^{\infty} b_l \psi_{k-l} = x_k. \tag{6.3}$$

Using this we get for $\Psi(D) = \mathcal{D}(\{\psi_k\})$,

$$\Psi(D) = \frac{X(D)}{B(D)} \overset{(a)}{=} \frac{X(D)}{P_c(D)},$$

where (a) follows due to the fact that we are using the settings of the ZF-DFE given in (6.1). Now, suppose instead of $\{x_k\}$, we transmit $\{\psi_k\}$. Then, after matched filtering,

$$Y(D) = ||p||Q(D)\Psi(D) + Z(D).$$

Now, suppose we go through the feed-forward filter $W(D)$, then we get,

$$\begin{aligned} Y(D)W(D) &= ||p||W(D)Q(D)\Psi(D) + Z(D)W(D) \\ &= \frac{||p||\gamma_0 P_c(D) P_c^*(D^{-*})}{\gamma_0 ||p|| P_c^*(D^{-*})} \frac{X(D)}{P_c(D)} + \frac{Z(D)}{\gamma_0 ||p|| P_c^*(D^{-*})} \\ &= X(D) + \tilde{Z}(D) \end{aligned}$$

where the noise PSD is,

$$S_{\tilde{Z}}(D) = \frac{N_0 Q(D)}{\gamma_0^2 ||p||^2 P_c(D) P_c^*(D^{-*})} = \frac{N_0}{\gamma_0 ||p||^2}$$

Hence, we have removed ISI, and the noise is white! The basic problem with this first attempt is that the transmit power is boosted. To see this, if $S_x(D) = \mathcal{E}_x$ is white, then,

$$S_\Psi(D) = \frac{\mathcal{E}_x}{P_c(D) P_c^*(D^{-*})} = \frac{\mathcal{E}_x \gamma_0}{Q(D)}$$

The transmit power is then

$$\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} S_\Psi(e^{-j\omega T}) d\omega$$

which could be larger than $\mathcal{E}_x$. Moreover, $\Psi(D)$ is no longer a white process, *i.e.,* there is memory in transmitted symbols. A simple trick eliminates the power problem almost completely.

Modulo operator: $\Gamma_M(x)$ is a non-linear function defined on a M-ary PAM input constellation with uniform spacing of d such that

$$\Gamma_M(x) = x - Md \lfloor \frac{x + \frac{Md}{2}}{Md} \rfloor$$

where $\lfloor y \rfloor$ means the largest integer that is less than or equal to y (lower ceil operation). The modulo operator is illustrated in Figure 6.2. Therefore the modulo operator $\Gamma_M(\cdot)$ maps all points $x \in \mathbb{R}$ back to the interval $[-\frac{Md}{2}, \frac{Md}{2}]$. This is done by first tiling the real line with intervals of size M. Then the intuitive explanation of the operator is as follows. We map any point $x \in \mathbb{R}$ by observing its relative

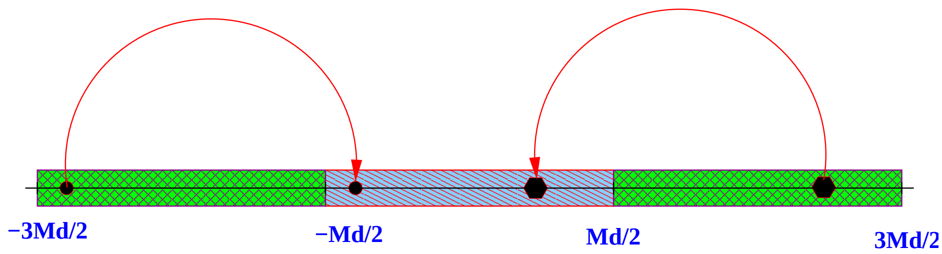


Figure 6.2: Illustration if the one-dimensional modulo operator.

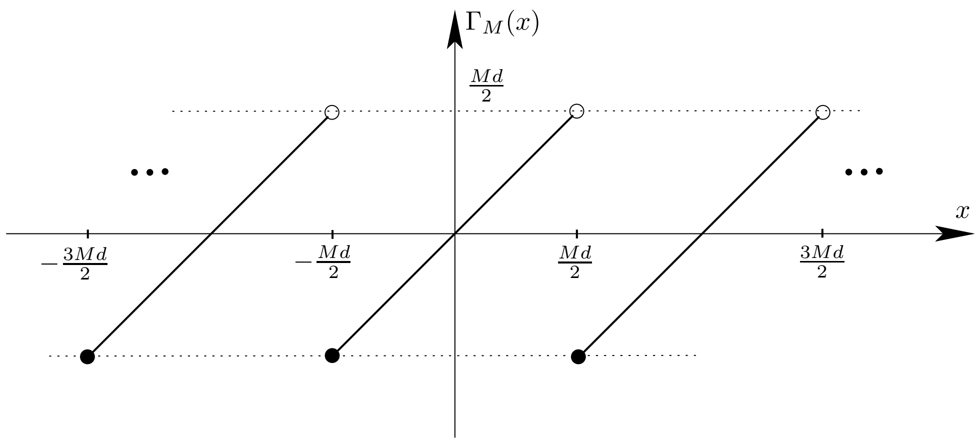


Figure 6.3: The modulo function.

position in the corresponding interval and mapping it back to the corresponding position in the interval $[-\frac{Md}{2}, \frac{Md}{2}]$ as shown in Figure 6.2.

Properties of modulo operator:

$$\begin{aligned} \Gamma_M(x+y) &= \Gamma_M(x) \oplus_M \Gamma_M(y) \\ \Gamma_M(x-y) &= \Gamma_M(x) \ominus_M \Gamma_M(y) \end{aligned}$$

where

$$\begin{aligned} a \oplus_M b &= \Gamma_M(a+b) \\ a \ominus_M b &= \Gamma_M(a-b) \end{aligned}$$

We denote $\oplus_M$ and $\ominus_M$ operators to mean addition and substraction modulo M. So, now, let us take our first idea and pass the output through a modulo operator in order to ensure the transmit power is controlled. From Figure 6.1, we see that,

$$\Psi(D) = X(D) + (1 - B(D))\tilde{X}(D)$$

where $B(D)$ is defined in (6.1). Therefore in time domain,

$$\psi_k = x_k - \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i}$$

where from Figure 6.1 we see that,

$$\tilde{x}_k = \Gamma_M(\psi_k) = \Gamma_M\left[x_k - \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i}\right]$$

Now, going back to our earlier calculation, with $W(D) = \frac{1}{\gamma_0 ||p|| P_c^*(D^{-*})}$ and $B(D) = P_c(D)$, we get

$$Y(D) = ||p||Q(D)\tilde{X}(D) + Z(D)$$

Hence after the feed forward filter,

$$\begin{aligned} R(D) = Y(D)W(D) &= \frac{||p||Q(D)}{\gamma_0 ||p|| P_c^*(D^{-*})}\tilde{X}(D) + \frac{Z(D)}{\gamma_0 ||p|| P_c^*(D^{-*})} \\ &= P_c(D)\tilde{X}(D) + \tilde{Z}(D) \end{aligned}$$

Therefore, in time domain,

$$r_k = \tilde{x}_k + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k$$

where the noise $\tilde{z}_k$ is white.
Now, suppose we do a modulo operation at the output, *i.e.*,

$$\begin{aligned} \Gamma_M(r_k) &= \Gamma_M\left[\tilde{x}_k + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k\right] \\ &= \Gamma_M\left[\Gamma_M(\psi_k) + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k\right] \\ &= \Gamma_M\left[\Gamma_M\left(x_k - \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i}\right) + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k\right] \\ &\overset{(a)}{=} \Gamma_M\left[x_k - \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \sum_{i=1}^{\infty} b_i \tilde{x}_{k-i} + \tilde{z}_k\right] \\ &= \Gamma_M\left[x_k + \tilde{z}_k\right] = x_k \oplus_M \Gamma_M(\tilde{z}_k) \end{aligned}$$

Here (a) follows because

$$\begin{aligned} \Gamma_M(\Gamma_M(a) + b) &= \Gamma_M(\Gamma_M(a)) \oplus_M \Gamma_M(b) \\ &= \Gamma_M(a) \oplus_M \Gamma_M(b) \\ &= \Gamma_M(a+b) \end{aligned}$$

The Tomlinson-Harashima precoder can be also adapted to the MMSE-DFE case, but we will not go into details here. Also, for complex symbols and multidimensional signals, the only change needed is in the definition of $\Gamma_M(x)$, the modulo-operator. All the arguments in the above stated case go through in such case as well with an appropriately defined modulo operator. Next, we address the transmit power of the precoder.
The original PAM constellation had an average energy of

$$\mathcal{E}_x = \frac{M^2 - 1}{12} d^2$$

For $\tilde{x}_k$ uniformly distributed in $[-Md/2, Md/2]$ the power is approximately $\frac{M^2 d^2}{12}$, which results in a power increase of $\frac{M^2}{M^2-1}$ in the transmission which implies a small increase in power that vanishes for large M. Note that this is an approximate calculation and not a formal result for finite M. A more complete calculation is beyond the scope of this class.

6.2 Multicarrier Transmission (OFDM)

Now, we have studied various methods of dealing with inter-symbol-interference. The main techniques were receiver driver, *i.e.,* the transmission was as if we had an AWGN channel, but the receiver took the ISI into account. One exception was the Tomlinson-Harashima precoder, which did change the transmission scheme. However it required knowledge of the channel of the transmitter. Next we are going to study a transmission and reception scheme that utilizes a characteristic of the channel (being a LTI channel) rather than the specifics of the channel response.

6.2.1 Fourier eigenbasis of LTI channels

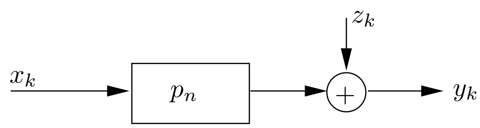


Figure 6.4: Transmission over LTI channel.

The basic idea is that the Fourier basis is the eigenbasis for LTI channels. In Figure 6.4, let $X(\omega) = \delta(\omega - \omega_0)$, then

$$X(\omega)P(\omega) = P(\omega)\delta(\omega - \omega_0) = P(\omega_0)\delta(\omega - \omega_0)$$

i.e.,

$$x(k) = e^{-j\omega_0 k} \Rightarrow x(k) * p(k) = P(\omega_0)e^{-j\omega_0 k} \tag{6.4}$$

Thus the output is just a scaled version of the input, in other words the Fourier basis is the eigenbasis for LTI channels. This property only utilizes the LTI nature of the channel and *not* any specific form of channel impulse response. This canonical orthogonal decomposition property is the central idea behind the transmission strategies discussed in this section.

6.2.2 Orthogonal Frequency Division Multiplexing (OFDM)

The idea in OFDM (Orthogonal Frequency Division Multiplexing) is to utilize this eigen-property of LTI channels. Let us use the modulation studied in Chapter 2, and therefore the cascade of the channel along with the basis function is defined as in Section 4.2. We use the same notation from there to denote the equivalent discrete time ISI channel as $\{p_n\}$. Therefore if we transmit (see Figure 6.5)

$$x_k = \sum_{n=0}^{N-1} X(n)e^{-j\omega_n k},$$

using the eigenproperty shown in (6.4), we see that the output y_k is

$$y_k = \sum_{n=0}^{N-1} X(n)P(\omega_n)e^{-j\omega_n k} + z_k$$

Now, we convolve the output using filters tuned to specific frequencies, $\omega_0, ..., \omega_{N-1}$.
Then we could just pick out the different frequency components of y_k and observe the noisy versions of what was transmitted, *i.e.,*

$$Y(n) = P(\omega_n)X(n) + Z(n), \;\; n = 0, \ldots, N-1,$$

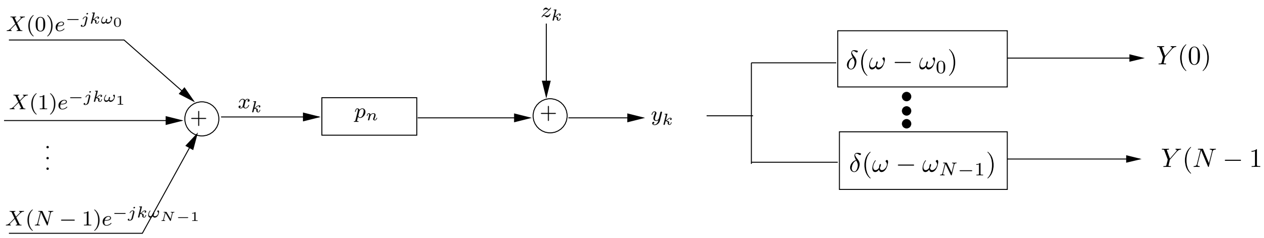


Figure 6.5: Decomposition of LTI channels using sinusoids.

where $Y(n)$ is the received signal at frequency ω_n, and $Z(n)$ is the corresponding noise at ω_n. Therefore we get a set of parallel channels, each of which is memoryless. Hence we have decomposed an ISI channel into a set of memoryless channels which are much easier to work with. However, in doing this we needed infinite length blocks. In reality we only have finite transmission blocks, and therefore a natural question to ask is whether this creates a roadblock to this approach. This is the topic we study next. Note that since we send information using multiple sinusoid "carriers" this idea is also called *multicarrier transmission.* Since we are using discrete spacing of the carriers the terminology commonly used is *Discrete Multi-Tone* or DMT for short.

Cyclic prefix

We now use an additional assumption on the channel response, *i.e.*, that it is a finite impulse response. A similar assumption was used in Section 5.7 while considering finite length equalizers. In fact we use the same setup as shown in Figure 6.6 for an oversampled receiver.
Consider the model given in (5.47)

$$\begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-N-\nu+1} \end{bmatrix} + \begin{bmatrix} z_k \\ z_{k-1} \\ \vdots \\ z_{k-N+1} \end{bmatrix} \tag{6.5}$$

Here, the output

$$\mathbf{y}_k = \begin{bmatrix} y_k(0) \\ \vdots \\ y_k(L-1) \end{bmatrix}$$

is the output of the "oversampler" and anti-aliasing filter, as shown in Figure 6.6. Also the pulse response is assumed finite, *i.e.,* of length νT. Now the idea is that for a block of N (vector) received samples[1], it is actually affected by $N+\nu$ input symbols, $\{x_k,\dots,x_{k-N-\nu+1}\}$ (see Figure 6.7).
Clearly the receiver symbols $\{y_{k-N},\dots,y_{k-N-\nu+1}\}$ depend on $\{x_{k-N},...,x_{k-N-\nu+1}\}$ which also affect the observation frame shown in (6.5). Therefore, in a way we are "discarding" those observations. Now, let us create a dependency between $\{x_{k-N},\dots,x_{k-N-\nu+1}\}$ and $\{x_k,\dots,x_{k-N+1}\}$ (see Figure 6.8). Therefore we are only sending N information symbols in a block of $N+\nu$ symbols.
Let us create a prefix $\{x_{k-N-\nu+1},...,x_{k-N+1}\}$

$$x_{k-N-i} = x_{k-i}, \quad i=0,\dots,\nu-1 \tag{6.6}$$

[1]For simplicity of notation we will sometimes use $L=1$, but the principle is identical in the more general (and correct) case.

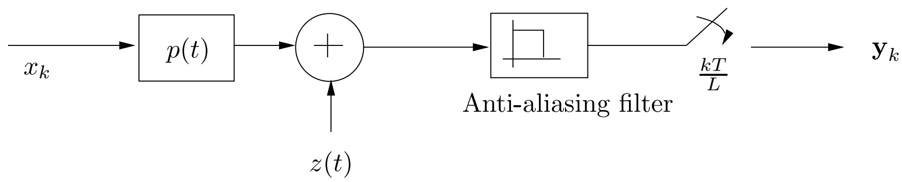


Figure 6.6: Set-up for (6.5).

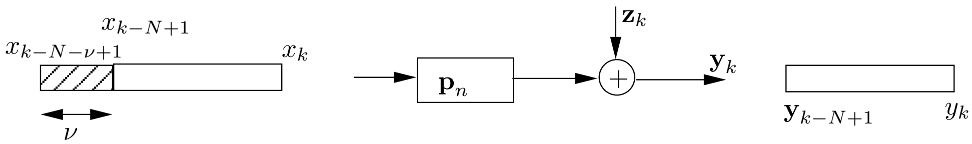


Figure 6.7: Output block $\{\mathbf{y}_k, ..., \mathbf{y}_{k-N+1}\}$ affected by inputs $\{x_k, ..., x_{k-N-\nu+1}\}$.

This implies that the frame looks "cyclic" it makes the data circularly shifted.

Claim 6.2.1. *Using the prefix (6.6) in (6.5), we get,*

$$\underbrace{\begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix}}_{\tilde{\mathbf{Y}}_k} = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu \\ \mathbf{p}_\nu & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{p}_0 & \dots & \mathbf{p}_{\nu-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_\nu & \mathbf{0} & \dots & \mathbf{0} & \mathbf{p}_0 \end{bmatrix}}_{\tilde{\mathbf{P}}} \underbrace{\begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-N+1} \end{bmatrix}}_{\tilde{\mathbf{X}}_k} + \underbrace{\begin{bmatrix} z_k \\ z_{k-1} \\ \vdots \\ z_{k-N+1} \end{bmatrix}}_{\tilde{\mathbf{Z}}_k} \tag{6.7}$$

or we get

$$\tilde{\mathbf{Y}}_k = \tilde{\mathbf{P}}\tilde{\mathbf{X}}_k + \tilde{\mathbf{Z}}_k \tag{6.8}$$

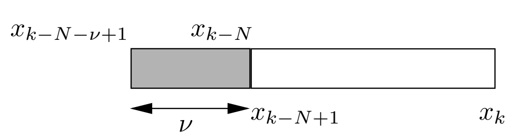


Figure 6.8: A transmission block of $N + \nu$ symbols.

Proof: First, let us rewrite (6.5) without the noise term as,

$$\begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-N+1} \\ \vdots \\ x_{k-N-\nu+1} \end{bmatrix} \tag{6.9}$$

Now, let us use the prefix defined in (6.6) which is,

$$x_{k-N-l} = x_{k-l}, \quad l = 0, \dots, \nu - 1$$

therefore the noiseless output at time $k-N+1, ..., k+\nu-N+1$ (which get affected by the prefix symbols) are given by.

$$\begin{aligned} \mathbf{y}_{k+\nu-N+1} &= \mathbf{p}_0 x_{k+\nu-N+1} + \cdots + \mathbf{p}_\nu x_{k-N+1} \\ \mathbf{y}_{k+\nu-N} &= \mathbf{p}_0 x_{k-\nu-N} + \cdots + \mathbf{p}_{\nu-1} x_{k-N+1} + \mathbf{p}_\nu \underbrace{x_{k-N}}_{x_k} \\ &\vdots \\ \mathbf{y}_{k-N+1} &= \mathbf{p}_0 x_{k-N+1} + \mathbf{p}_1 \underbrace{x_{k-N}}_{x_k} + \dots + \mathbf{p}_{\nu-1} \underbrace{x_{k-N-\nu+2}}_{x_{k-(\nu-2)}} + \mathbf{p}_\nu \underbrace{x_{k-N-\nu+1}}_{x_{k-(\nu-1)}} \end{aligned}$$

Hence inserting this in equation (6.9) we get the noiseless output as

$$\begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k+\nu-N+1} \\ \mathbf{y}_{k+\nu-N} \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{p}_0 & \mathbf{p}_1 & \dots & \mathbf{p}_\nu \\ \mathbf{p}_\nu & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{p}_0 & \dots & \mathbf{p}_{\nu-1} \\ & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots \\ \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_\nu & \mathbf{0} & \dots & \mathbf{0} & \mathbf{p}_0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k+\nu-N+1} \\ \vdots \\ x_{k-N+1} \end{bmatrix} \tag{6.10}$$

Therefore after re-inserting the noise we get (6.7) as claimed.

□

Now, let us look at $L = 1$, *i.e.,* each of the channel taps $\{\mathbf{p}_i\}_{i=0}^{\nu}$ is a scalar rather than a vector.

Theorem 6.2.1 (Decomposition of circulant matrices)**.** *A ($N \times N$) matrix is circulant if it has the following structure.*

$$\mathbf{C} = \begin{bmatrix} p_0 & p_1 & \dots & p_\nu & 0 & \dots & 0 \\ 0 & p_0 & p_1 & \dots & p_\nu & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p_0 & p_1 & \dots & p_\nu \\ p_\nu & 0 & \dots & 0 & p_0 & \dots & p_{\nu-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & \dots & p_\nu & 0 & \dots & p_0 \end{bmatrix}$$

A circulant matrix has an eigen-decomposition

$$\mathbf{C} = \mathbf{F}^* \mathbf{\Lambda} \mathbf{F}$$

where

$$(\mathbf{F})_{p,q} = \frac{1}{\sqrt{N}} \exp\left\{-j\frac{2\pi}{N}(p-1)(q-1)\right\}$$

and $\mathbf{\Lambda} = diag(d_l)$ *where* d_l *is*

$$d_l = \sum_{m=0}^{\nu} p_m e^{-j\frac{2\pi}{N}ml}$$

That is,

$$\mathbf{\Lambda} = \begin{bmatrix} d_0 & & \mathbf{0} \\ & \vdots & \\ \mathbf{0} & & d_{N-1} \end{bmatrix}$$

■

Note: The matrix $\mathbf{F}$ as one would recognize is the DFT matrix and $\mathbf{\Lambda}$ contains on its diagonal the Fourier transform of the channel taps.
Now, let us use Theorem 6.2.1 in the basic OFDM equation (6.8) which is,

$$\tilde{\mathbf{Y}}_k = \tilde{\mathbf{P}}\tilde{\mathbf{X}}_k + \tilde{\mathbf{Z}}_k$$

Now, let us define

$$\begin{aligned} \mathbf{X}_k &= \mathbf{F}\tilde{\mathbf{X}}_k \\ \mathbf{Y}_k &= \mathbf{F}\tilde{\mathbf{Y}}_k \\ \mathbf{Z}_k &= \mathbf{F}\tilde{\mathbf{Z}}_k \end{aligned}$$

Then using Theorem 6.2.1 we have,

$$\mathbf{Y}_k = \mathbf{\Lambda}\mathbf{X}_k + \mathbf{Z}_k$$

since the DFT matrix is unitary *i.e.,*

$$\mathbf{F}^*\mathbf{F} = \mathbf{I}$$

Hence

$$\mathbf{Y}_k(l) = d_l\mathbf{X}_k(l) + \mathbf{Z}_k(l), \quad l = 0, \dots, N-1 \tag{6.11}$$

Therefore we have created ISI-free channel! Basically to do this, we have used the eigen property of LTI channels as well as the finite impulse response property of the channel. Note that for arbitrary L, (6.11) just becomes a vector relationship in the following way.

$$\mathbf{Y}_k(l) = \mathbf{D}_l\mathbf{X}_k(l) + \mathbf{Z}_k(l), l = 0, \dots, N-1$$

Here

$$\mathbf{D}_l = \begin{bmatrix} \sum_{m=0}^{\nu} p_0(m)e^{-j\frac{2\pi}{N}ml} \\ \vdots \\ \sum_{m=0}^{\nu} p_{L-1}(m)e^{-j\frac{2\pi}{N}ml} \end{bmatrix} = \sum_{m=0}^{\nu} \mathbf{p}(m)e^{-j\frac{2\pi}{N}ml}.$$

6.2.3 Frequency Domain Equalizer (FEQ)

Now, the relationship given in (6.11) for $L = 1$, forms the basis of the receiver in a manner similar to (5.45) for finite length equalizers. Given that we have no interference between the symbols $\{\mathbf{X}_n(l)\}$ the MMSE receiver is quite simple. Since we have transmitted symbols in additive (white) noise, the MMSE estimator is just.

$$\hat{\mathbf{X}}_l(k) = \frac{d_l^*}{|d_l|^2 + \sigma^2/\tilde{\mathcal{E}}_x}\mathbf{Y}_k(l)$$

where $\tilde{\mathcal{E}}_x$ is the energy on every symbol. Since we are doing N successive transmissions, and $\mathbf{F}$ is a unitary transform, typically $\tilde{\mathcal{E}}_x = \mathcal{E}_x$, the energy of originally intended symbol. The above FEQ is equivalent to the MMSE-LE applied in the frequency domain.

6.2.4 Alternate derivation of OFDM

Consider a periodic sequence $\{x_n\}$ of period N, given by

$$x_{k-rN} = x_k \ , \ r \in \mathbb{Z} \ , k \in \{0, \dots, N-1\}$$

Therefore the Fourier transform of $\{x_k\}$ is (basically Fourier series coefficients) is given by

$$X(\omega) = \sum_{n=0}^{N-1} \mathcal{X}_n \delta(\omega - \frac{2\pi}{N} n)$$

Now, let us pass this periodic sequence through a LTI channel as shown in Figure 6.9.
The output $Y(\omega)$ is,

$$Y(\omega) = P(\omega)X(\omega) = P(\omega)\sum_{n=0}^{N-1} \mathcal{X}_n \delta(\omega - \frac{2\pi}{N} n) = \sum_{n=0}^{N-1} \mathcal{X}_n P(\frac{2\pi}{N} n)\delta(\omega - \frac{2\pi}{N} n) \tag{6.12}$$

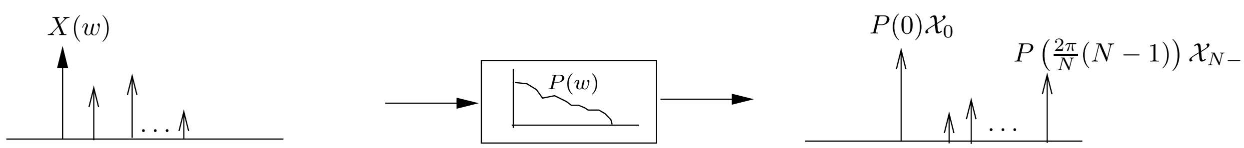


Figure 6.9: Transmission of periodic sequence through LTI channel.

Therefore, $\{y_k\}$ is also periodic with period N, *i.e.,*

$$y_{k+rN} = y_k \ , \ r \in \mathbb{Z} \ , k \in \{0, \dots, N-1\}$$

Hence, just N samples of $\{y_k\}_{k=0}^{N-1}$ are enough to reconstruct the sequence.
Therefore, the Fourier coefficients of $Y(\omega)$ are

$$Y(\omega) = \sum_{n=0}^{N-1} \mathcal{Y}_n \delta(\omega - \frac{2\pi}{N} n) \tag{6.13}$$

From equations (6.12) and (6.13), we obtain,

$$\mathcal{Y}_n = P\left(\frac{2\pi n}{N}\right)\mathcal{X}_n, \quad n = 0, \dots, N-1. \tag{6.14}$$

Hence, if we transmit periodic sequences, $\{x_n\}$, over an LTI channel, we can get a "straight-wire" relationship between $\{\mathcal{X}_n\}_{n=0}^{N-1}$ and $\{\mathcal{Y}_n\}_{n=0}^{N-1}$.
The basic trick in DMT/OFDM is to "simulate" an infinite periodic sequence without actually having to do so. This *critically* uses the FIR property of the channel. Since in the periodic sequence $\{y_k\}$, the N samples are enough to recreate the periodic sequence, can we create a non-periodic transmission strategy that produces the *same* output? Therefore if we create an input that produces N samples of y_k which is the *same* as if the input were periodic, then we are done.
Now, if $P(\omega)$ corresponds to FIR channel of memory ν, *i.e.*, has impulse response $p_0, ..., p_\nu$, then the noiseless output is

$$y_k = \sum_{n=0}^{\nu} p_n x_{k-n}.$$

For $y_0, \ldots, y_{N-1}$ to look the same as the first N samples of the periodic sequence, we just need to *prefix* the sequence $\{x_k\}$ *i.e.,* see Figure 6.10.

	Information
$x_{N-\nu} \ldots x_N$	$x_0 \ldots x_{N-1}$
Prefix	

Figure 6.10: The cyclic prefix.

Therefore, by using the cyclic prefix (for simplicity assume $N > \nu$), we get

$$\begin{aligned}
y_0 &= p_0 x_0 + p_1 \underbrace{x_{N-1}}_{x_{-1}} + p_2 \underbrace{x_{N-2}}_{x_{-2}} + \ldots + p_\nu \underbrace{x_{N-\nu}}_{x_{-\nu}} \\
y_1 &= p_0 x_1 + p_1 x_0 + p_2 x_{N-1} + \ldots + p_\nu \underbrace{x_{N-\nu+1}}_{x_{-\nu+1}} \\
&\vdots \\
y_{\nu-1} &= p_0 x_{\nu-1} + p_1 x_{\nu-2} + \ldots + p_{\nu-1} x_0 + p_\nu x_{N-1} \\
y_\nu &= p_0 x_\nu + p_1 x_{\nu-1} + \ldots + p_{\nu-1} x_1 + p_\nu x_0 \\
&\vdots \\
y_{N-1} &= p_0 x_{N-1} + p_1 x_{N-2} + \ldots + p_\nu x_{N-1-\nu}
\end{aligned} \tag{6.15}$$

Therefore, the output $y_0, \ldots, y_{N-1}$ is *identical* to that we would have obtained from a periodic sequence. Hence, we have "tricked" the FIR channel into thinking that we did send a periodic sequence and get the "straight-wire" channel of equation (6.14). This idea has been illustrated for just a transmission block from time $k = 0, ..., N-1$. However, in Section 6.2.2 we had transmission block which ended at an arbitrary time index k and hence the time indices look different. However, if we shifted our blocks to end at k instead of $N-1$, then we would get exactly the same result as given in (6.11). Therefore the relationship in (6.14) is the noiseless version of the relationship in (6.11). Therefore we have derived the OFDM by an alternate methodology which also gives us the interpretation for why it works.
Notes:

1. We of course cannot have a time-limited and band-limited channels. This is an approximation, and this truncation would in principle cause aliasing. However, the choice of ν can make this error as small as we need.

2. By the trick shown, we essentially "sample" the channel in the frequency domain,*i.e.,* $\left\{P\left(\frac{2\pi}{N}n\right)\right\}_{n=0}^{N-1}$. This is correct for periodic sequences, but with a-periodic sequences, such a sampling could incur a loss, which is the same as the "aliasing" error in 1) above.

3. The decomposition of the channel into parallel channels through the cyclic prefix trick did *not* need the transmitter to know the channel impulse response. All that was used was

   - The channel was LTI
   - It had a finite response of length $\nu + 1$

   This property makes the decomposition canonical for FIR LTI channels.

The interpretation through periodic sequences leads us to the use of circular convolution. For a given N, the circular convolution between two sequences $u_0, ..., u_{N-1}$ and $v_0, ..., v_{N-1}$ is defined as

$$u_k \otimes_N v_k = \sum_{n=0}^{N-1} u_n v_{(k-n)_N}, \quad k = 0, ..., N-1 \tag{6.16}$$

where $(k-n)_N$ is the modulo operator *i.e.,* $(k-n)_N = r$ if $k - n = qN + r$, for some $q \in \mathbb{Z}$. Therefore in our case if $(k-n) < 0$ then

$$(k-n)_N = N - (k-n).$$

Hence we can interpret the relationship in (6.15) by a circular convolution, *i.e.,*

$$y_k = p_k \otimes_N x_k.$$

This is also true in the relationship in (6.10) where

$$\mathbf{y}_k = \mathbf{p}_k \otimes_N x_k$$

Thus the operations in DMT/OFDM can also be interpreted in terms of circular convolutions. This interpretation is explicit in our derivation using periodic sequences. Note that circular convolution like periodic convolution is a commutative operation.

6.2.5 Successive Block Transmission

Note that the development till now has been on a single block of transmission. What happens for continuous transmissions? This is illustrated in Figure 6.13. For continuous transmission, the idea goes through again in the following manner.

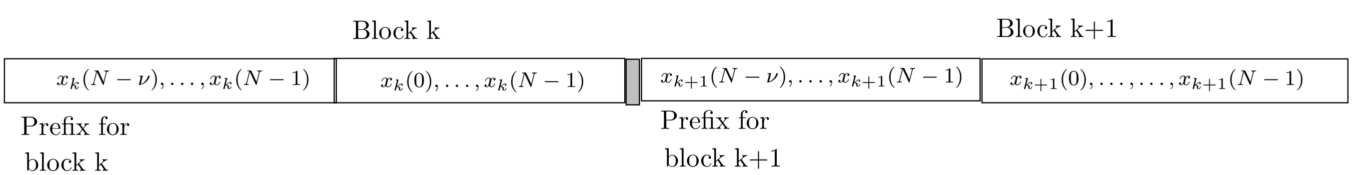


Figure 6.11: Transmitted sequence for OFDM blocks.

We define the following quantities (see Figures 6.11 and 6.12) in relation to the single frame transmission notation given in (6.7). We denote the frame of $N + \nu$ symbols transmitted in block k as

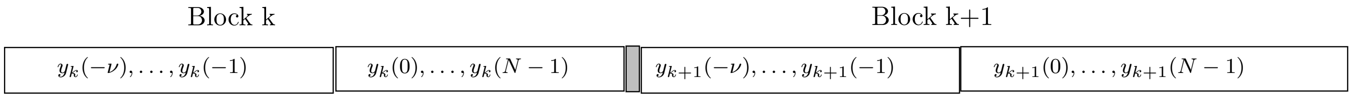


Figure 6.12: Received sequence for OFDM blocks.

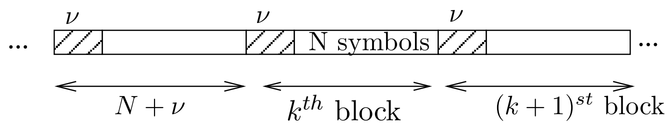


Figure 6.13: Continuous transmission blocks.

$\{x_k(-\nu), ..., x_k(-1), x_k(0)..., x_k(N-1)\}$ which along with the cyclic prefix relationship in (6.6) we see that the transmitted frame is $\{x_k(N-\nu), ..., x_k(N-1), x_k(0), ..., x_k(N-1)\}$.
The received frame of $N+\nu$ symbols for block k is denoted by, $\{y_k(-\nu), ..., y_k(-1), y_k(0), ..., y_k(n-1)\}$.
Now for *each* block of transmission, it is clear that $\{y_k(-\nu), ..., y_k(-1)\}$ are influenced by symbols transmitted in the previous frame (*i.e.,* $x_{k-1}(l), l = 0, ..., N-1$). Therefore, we discard $y_k(-\nu), ..., y_k(-1)$ and only use $y_k(0), ..., y_k(N-1)$. Hence,

$$y_{k+1}(0) = p_0 x_{k+1}(0) + p_1 \underbrace{x_{k+1}(N-1)}_{x_{k+1}(-1)} + ... + p_\nu \underbrace{x_{k+1}(N-\nu)}_{x_{k+1}(-\nu)}$$

where due to the cyclic prefix the interference from the previous transmitted block (block k) has been eliminated. Therefore the prefix has eliminated the interblock interference at the cost of added redundancy of ν symbols per block. Hence the redundancy rate is $\frac{\nu}{N+\nu}$ per transmitted symbol.
Note that if we examined $y_{k+1}(-1)$, we would have dependencies from the previous transmitted block (block k) and this is the reason for discarding it. Therefore the prefix given in (6.6) serves another purpose, *i.e.,* it prevents successive blocks from interfering with each other. This function causes it to be also called the "guard" interval.
The overall block diagram implementation of DMT is shown in Figure 6.14.

Disadvantages of OFDM

The OFDM technique seems like a panacea and solves our ISI channel problem. However, it does have some disadvantage like longer transmit symbols, larger peak-to-average ratio, sensitivity to mismatch and synchronization errors. So, in practice, it depends on the real gain in system performance to make the appropriate choice between time-domain equalization and OFDM. Sometimes a combination of the two might be appropriate.

6.3 Channel Estimation

Up to now, the main assumption is that the transmission channel is known at the receiver. All the algorithms and techniques we have seen critically utilize this fact. A natural question is how we have such knowledge. The basic idea in channel estimation is to send known information symbols and estimate the channel. The technique is a dual of estimating information symbols while sending it through a known channel. Let us re-examine the discrete-time model given in (5.46) and for simplicity consider $L = 1$.

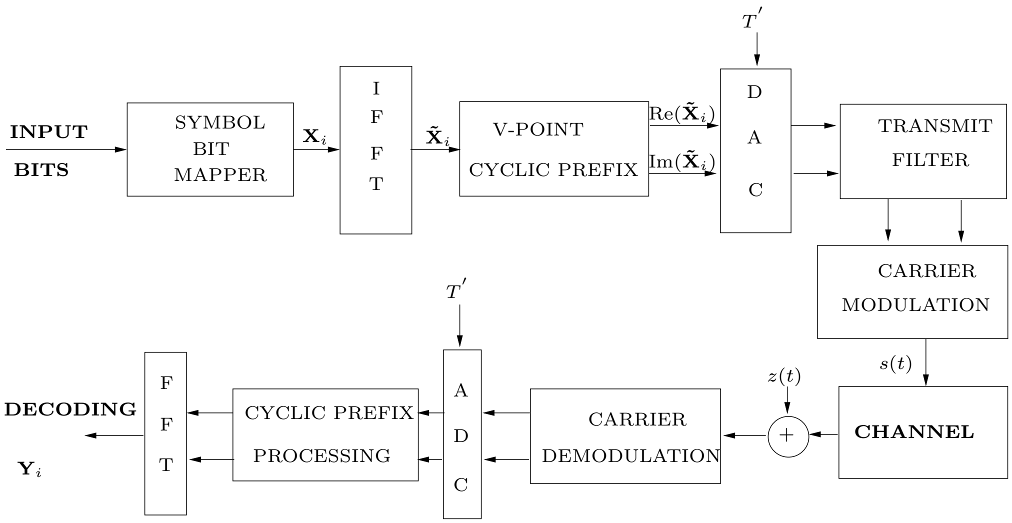


Figure 6.14: Block diagram for OFDM.

Therefore, we can re-write it as

$$y_k = \sum_{m=0}^{\nu} p_m x_{k-m} + z_k,$$

and in vector form it is,

$$\underbrace{\begin{bmatrix} y_k \\ \vdots \\ y_{k-N_T+1} \end{bmatrix}}_{\mathcal{Y}} = \underbrace{\begin{bmatrix} x_k & x_{k-1} & \dots & x_{k-\nu} \\ x_{k-1} & x_{k-2} & \dots & x_{k-\nu-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{k-N_+1} & x_{k-N_T} & \dots & x_{k-N_T+1-\nu} \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_\nu \end{bmatrix}}_{\mathbf{p}} + \underbrace{\begin{bmatrix} z_k \\ \vdots \\ \vdots \\ z_{k-N_T+1} \end{bmatrix}}_{\mathbf{z}} \tag{6.17}$$

where N_T = length of training. The problem is now of estimating $\mathbf{p}$ from $\mathcal{Y}$ when $\{x_k, \dots, x_{k-N_T+1-\nu}\}$ are known symbols. Thus in compact form, (6.17) is written as

$$\mathcal{Y} = \mathcal{X}\mathbf{p} + \mathbf{z} \tag{6.18}$$

A criterion to estimate $\mathbf{p}$ from $\mathcal{Y}$ is

$$\hat{\mathbf{p}}_{opt} = \arg\min_{\mathbf{p}} ||\mathcal{Y} - \mathcal{X}\mathbf{p}||^2 \tag{6.19}$$

This can be written in more detail as,

$$\arg\min_{\mathbf{p}} \sum_{t=0}^{N_T-1} \left| y_{k-t} - \sum_{m=0}^{\nu} p_m x_{k-t-m} \right|^2 = \hat{\mathbf{p}}_{opt}$$

Note that this criterion does *not* involve any expectation ($\mathbb{E}[\cdot]$) operation. This is because we are assuming the channel to be an unknown constant and not a stochastic quantity. The criterion given in (6.19) is called the *deterministic least squares* criterion. As we shall see below, several of the geometric interpretations

we had for the MMSE criterion (*stochastic least squares*) in Section 5.1 also carries through here as well. We will examine a relationship between deterministic and stochastic least squares in Section 6.3.2.
We now know derive the orthogonality principle appropriate for the deterministic least squares problem (see also Figure 6.15). Let us write the cost function given in (6.19) as,

$$\begin{aligned}\mathcal{J}(p) &= ||\mathcal{Y} - \mathcal{X}\mathbf{p}||^2 \\ &= \mathcal{Y}^*\mathcal{Y} + \mathbf{p}^*\mathcal{X}^*\mathcal{X}\mathbf{p} - \mathcal{Y}^*\mathcal{X}\mathbf{p} - \mathbf{p}^*\mathcal{X}^*\mathcal{Y}\end{aligned}$$

By differentiation one gets

$$\mathcal{X}^*\mathcal{X}\hat{\mathbf{p}}_{opt} = \mathcal{X}^*\mathcal{Y} \tag{6.20}$$

which in the terminology of least squares is the *normal equation.* If $\mathcal{X}^*\mathcal{X}$ is invertible then

$$\hat{\mathbf{p}}_{opt} = (\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*\mathcal{Y} \tag{6.21}$$

Note that the normal equation (6.20) implies the orthogonality principle, *i.e.,*

$$\begin{aligned}e_{opt} &= \mathcal{Y} - \mathcal{X}\hat{\mathbf{p}}_{opt} \\ &= \mathcal{Y} - \mathcal{X}(\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*\mathcal{Y} \\ \Rightarrow \mathcal{X}^*e_{opt} &= \mathcal{X}^*\mathcal{Y} - (\mathcal{X}^*\mathcal{X})(\mathcal{X}\mathcal{X}^*)^{-1}\mathcal{X}^*\mathcal{Y} \\ &= 0 \qquad (6.22)\end{aligned}$$

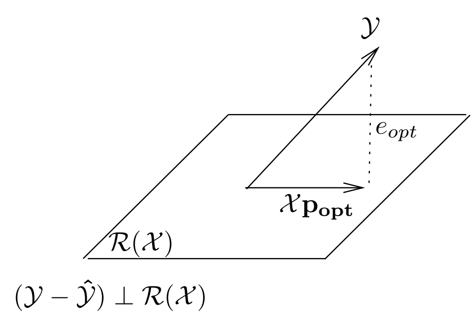


Figure 6.15: Geometry of the deterministic least squares problem.

Hence $\mathbf{e}_{opt} \perp \mathcal{X}$, is the orthogonality principle in this context (see also Figure 6.15).
Now, let us use this result in the model of interest, *i.e.,* (6.18). Therefore, we get

$$\hat{\mathbf{p}}_{opt} = (\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*[\mathcal{X}\mathbf{p} + \mathbf{z}] = \mathbf{p} + (\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*\mathbf{z} \tag{6.23}$$

The estimation error for criterion (6.19) is therefore

$$e_{opt} = \mathcal{Y} - \mathcal{X}(\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*\mathcal{Y} = \left[\mathbf{I} - \mathcal{X}(\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*\right]\mathcal{Y}$$

Now, we can find the least squares error norm as

$$\begin{aligned}\| e_{opt} \|^2 &= \| \mathcal{Y} - \mathcal{X}\hat{\mathbf{p}}_{opt} \|^2 = \mathcal{Y}^*\mathbf{e}_{opt} - \hat{\mathbf{p}}_{opt}\mathcal{X}^*\mathbf{e}_{opt} \\ &\overset{(a)}{=} \mathcal{Y}^*\mathbf{e}_{opt} = \mathcal{Y}^*\mathcal{Y} - \mathcal{Y}^*\mathcal{X}\hat{\mathbf{p}}_{opt} \\ &= \mathcal{Y}^*\mathcal{Y} - (\mathcal{X}^*\mathcal{Y})^*\hat{\mathbf{p}}_{opt} \\ &\overset{(b)}{=} \mathcal{Y}^*\mathcal{Y} - (\mathcal{X}^*\mathcal{X}\hat{\mathbf{p}}_{opt})^*\hat{\mathbf{p}}_{opt} \\ &= \mathcal{Y}^*\mathcal{Y} - \| \mathcal{X}\hat{\mathbf{p}}_{opt} \|^2\end{aligned}$$

where (a) follows due to orthogonality (6.22) and (b) follows due to the normal equation (6.20). Therefore we get

$$\| e_{opt} \|^2 = \mathcal{Y}^*\mathcal{Y} - p_{opt}^*(\mathcal{X}^*\mathcal{X})\hat{\mathbf{p}}_{opt} \tag{6.24}$$

which is a Pythagorean relationship (see Figure 6.15) similar to Theorem 5.1.3 in Section 5.1.4.

6.3.1 Training sequence design

We have further control on the channel estimation problem by appropriately choosing the known ("training") symbols that are sent. In this section we explore some properties we desire for designing training sequences.

Now, using (6.23), the error in the estimate $\hat{\mathbf{p}}_{opt}$ is,

$$\begin{aligned} \hat{\mathbf{p}}_{opt} - \mathbf{p} &= (\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*[\mathcal{X}\mathbf{p} + \mathbf{z}] - \mathbf{p} \qquad (6.25)\\ &= (\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*\mathbf{z} \end{aligned}$$

Hence,

$$\mathbb{E}_{\mathbf{z}}[||\hat{\mathbf{p}}_{opt} - \mathbf{p}||^2] = N_0 trace(\mathcal{X}^*\mathcal{X})^{-1},$$

where we have taken the $\mathbb{E}[\cdot]$ operation over the noise $\mathbf{z}$, and used $\mathbb{E}[\mathbf{z}\mathbf{z}^*] = N_0\mathbf{I}$. Therefore, the "quality" of the estimate is determined by $trace[(\mathcal{X}^*\mathcal{X})^{-1}]$. The "optimal" training design is to choose the training symbols so as to minimize $trace[(\mathcal{X}^*\mathcal{X})^{-1}]$ for a given power constraint.

Problem:

$$\begin{aligned} \text{Minimize} \quad & trace[(\mathcal{X}^*\mathcal{X})^{-1}] \\ \text{subject to} \quad & \frac{1}{N_T+\nu}\sum_{i=0}^{N_T+\nu-1}|x_{k-i}|^2 \le \mathcal{E}_x \qquad (6.26)\\ \text{or} \quad & trace[\mathcal{X}^*\mathcal{X}] \le c \qquad (6.27) \end{aligned}$$

Facts:

1. **Unitary decomposition:** If $\mathbf{A} = \mathbf{A}^*$, then $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$ where $\mathbf{U}^*\mathbf{U} = \mathbf{I}$.
2. If $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$ and $\mathbf{U}\mathbf{U}^* = \mathbf{I}$ then, $\mathbf{A}^{-1} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^*$.

Theorem 6.3.1. *Consider the problem with constraint given in (6.27),* i.e.,

$$\begin{aligned} \textit{Minimize} \quad & trace[(\mathcal{X}^*\mathcal{X})^{-1}] \\ \textit{subject to} \quad & trace[\mathcal{X}^*\mathcal{X}] \le c \end{aligned} \tag{6.28}$$

Then optimal training sequence for this problem satisfies $\mathcal{X}^*\mathcal{X} = \frac{c}{\nu+1}\mathbf{I}$.

Proof: Let $\mathcal{X}^*\mathcal{X} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$

$$\begin{aligned} trace(\mathcal{X}^*\mathcal{X})^{-1} &= trace(\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^*) = trace(\mathbf{\Lambda}^{-1}\mathbf{U}^*\mathbf{U}) \\ &= trace(\mathbf{\Lambda}^{-1}) = \sum_{i=0}^{\nu}\frac{1}{\lambda_i} \end{aligned}$$

Similarly

$$trace(\mathcal{X}^*\mathcal{X}) = \sum_{i=0}^{\nu}\lambda_i$$

The problem with the constraint in (6.27) is then,

$$\text{Minimize } \sum \frac{1}{\lambda_i}$$

$$\text{s.t } \sum \lambda_i \leq c$$

Let us write the Lagrangian form for constrained optimization,

$$\begin{aligned} \mathcal{J}(\lambda) &= \sum \frac{1}{\lambda_i} - \gamma \sum \lambda_i \\ \frac{\partial \mathcal{J}}{\partial \lambda_i} &= -\frac{1}{\lambda_i^2} - \gamma = 0 \end{aligned}$$

Therefore we see that $\lambda_i^2 = -\frac{1}{\gamma}$ is a constant independent of index i. Therefore

$$\Lambda = -\frac{1}{\sqrt{\gamma}}\mathbf{I},$$

giving us

$$\mathcal{X}^*\mathcal{X} = -\frac{1}{\sqrt{\gamma}}\mathbf{I}.$$

Using this in (6.27) with equality we get

$$\mathcal{X}^*\mathcal{X} = \frac{c}{\nu + 1}\mathbf{I}$$

This was a toy problem in that we did not take the exact constraints into account (there are edge effects that mean that problem (6.26) and (6.27) are not equivalent). Also, typically we want $\{x_k\}$ to come from a particular constellation. That makes the training sequence optimization problem a discrete one, *i.e.,* a discrete optimization problem. However, this gives one an idea of what kind of training sequences we should attempt to design. Another way of interpreting this is that we want the training sequences to have low cross-correlation with shifted versions. This requirement on sequences also arises in CDMA wireless multiple access communications.

Now, let us compare the true constraint given in (6.26)

$$\frac{1}{N_T + \nu} \sum_{l=0}^{N_T+\nu-1} |x_{k-l}|^2 \leq \mathcal{E}_x$$

and the formal constraint (6.27) that we solved, *i.e.,*

$$tr\left[\mathcal{X}^*\mathcal{X}\right] \leq c.$$

In order to do this we can write,

$$\mathcal{X}^*\mathcal{X} = \begin{bmatrix} x_k & \dots & x_{k-\nu} \\ \vdots & & \vdots \\ x_{k-N_T+1} & \dots & x_{k-\nu-N_T+1} \end{bmatrix}^* \begin{bmatrix} x_k & \dots & x_{k-\nu} \\ \vdots & & \vdots \\ x_{k-N_T+1} & \dots & x_{k-\nu-N_T+1} \end{bmatrix}$$

Hence we get,

$$\begin{aligned} trace\,(\mathcal{X}^*\mathcal{X}) &= \sum_{l=0}^{\nu}\sum_{t=0}^{N_T-1}|x_{k-l-t}|^2 \qquad (6.29)\\ &= (\nu+1)\left[\sum_{l=\nu}^{N_T-1}|x_{k-l}|^2\right]+|x_k|^2+2|x_{k-1}|^2+\ldots+\nu|x_{k-\nu+1}|^2+\nu|x_{k-N_T}|^2\\ &\quad +\ldots+2|x_{k-\nu-N_T+2}|^2+|x_{k-\nu-N_T+1}|^2 \end{aligned}$$

Therefore we get,

$$\begin{aligned} trace\,(\mathcal{X}^*\mathcal{X}) &= (\nu+1)\left[\sum_{l=0}^{N_T+\nu-1}|x_{k-l}|^2\right]-\left\{\nu|x_k|^2+\ldots+|x_{k-\nu+1}|^2\right\}-\left\{|x_{k-N_T}|^2+\ldots+\nu|x_{k-N_T-\nu+1}|^2\right\}\\ &\le (\nu+1)(N_T+\nu)\mathcal{E}_x-\left\{\nu|x_k|^2+\ldots+|x_{k-\nu+1}|^2\right\}-\left\{|x_{k-N_T}|^2+\ldots+\nu|x_{k-N_T-\nu+1}|^2\right\}\\ &\approx (\nu+1)(N_T+\nu)\mathcal{E}_x \qquad (6.30) \end{aligned}$$

Therefore $c \approx (\nu+1)(N_T+\nu)\mathcal{E}_x$ if we ignore the edge effects.
Now if we use the "optimal" training sequence sa specified in Theorem 6.3.1, we get

$$(\mathcal{X}^*\mathcal{X}) = \frac{c}{\nu+1}\mathbf{I} = (N_T+\nu)\mathcal{E}_x\mathbf{I}$$

and applying this using (6.25) we have

$$\mathbb{E}(\mathbf{p}-\hat{\mathbf{p}}_{opt})(\mathbf{p}-\hat{\mathbf{p}}_{opt})^* = (\mathcal{X}^*\mathcal{X})^{-1}N_0.$$

These together give us,

$$\mathbb{E}(\mathbf{p}-\hat{\mathbf{p}}_{opt})(\mathbf{p}-\hat{\mathbf{p}}_{opt})^* = \frac{N_0}{(N_T+\nu)\mathcal{E}_x}\mathbf{I}$$

Therefore, for the optimal training sequences,

$$\mathbb{E}[\parallel \mathbf{p}-\hat{\mathbf{p}}_{opt}\parallel^2] = \frac{N_0\nu}{(N_T+\nu)\mathcal{E}_x} = \left[\frac{\frac{1}{SNR}}{1+\frac{N_T}{\nu}}\right].$$

Given this, the error in the estimate is made small by choosing $N_T >> \nu$, *i.e.*, the training length should be much larger then the number of parameters being estimated. If such a large N_T is chosen, then the assumption of "perfect" channel state information becomes quite reasonable.

Channel mismatch

Since the channel estimation error is non-zero, the use of channel estimate instead of true channel incurs a penality. Therefore we think that the output behaves as,

$$\mathcal{Y}_k = \sum_{m=0}^{\nu}\hat{\mathbf{p}}_{opt,m}x_{k-m} + \underbrace{\sum(\mathbf{p}_m-\hat{\mathbf{p}}_{opt,m})x_{k-m}+\mathbf{z}_k}_{\mathbf{Z}'_k}$$

If $\{x_k\}$ are assumed i.i.d then

$$\mathbb{E}\left[||\mathbf{Z}'_k||^2\right] = \mathcal{E}_x\mathbb{E}\left[||\hat{\mathbf{P}}_{opt}-\mathbf{P}||^2\right]+N_0L$$

Hence, if the estimation error is small, then the use of the estimated channel does not greatly affect performance. However, if either the training sequence length N_T, is small of the sequence was chosen poorly, the mismatch in the channel could adversely affect performance. For good system design, one tries to ensure that the channel mismatch is small. If N_T becomes large, it is clear that the mismatch asymptotically vanishes. In wireless channels (which we will encounter soon), there may be constraints on N_T causing non-negligible channel mismatch.

6.3.2 Relationship between stochastic and deterministic least squares

In this section we explore the relationship between deterministic and stochastic least squares. We do this using the model given in (6.18)

$$\mathcal{Y} = \mathcal{X}\mathbf{p} + \mathbf{z}$$

and make the assumption that $\mathbf{p}$ is a stochastic variable with

$$\mathbb{E}\mathbf{p}\mathbf{p}^* = \alpha\mathbf{I},$$

As before we assume that

$$\mathbb{E}\mathbf{z}\mathbf{z}^* = N_0\mathbf{I}.$$

Now we use a linear estimator

$$\hat{\mathbf{p}}_s = \mathbf{M}\mathcal{Y} \tag{6.31}$$

and we use the criterion

$$\hat{\mathbf{p}}_s = \arg\min_{\hat{\mathbf{p}}} \mathbb{E}[\| \mathbf{p} - \hat{\mathbf{p}} \|^2] \tag{6.32}$$

Using orthogonality principle (seen in Section 5.1) we have

$$\mathbf{p} - \hat{\mathbf{p}}_s \perp \mathcal{Y}$$

Hence we have,

$$\mathbb{E}\left[(\mathbf{p} - \mathbf{M}\mathcal{Y})\,\mathcal{Y}^*\right] \quad = \quad \mathbf{0}$$

giving us

$$\mathbf{M} \quad = \quad \mathbb{E}(\mathbf{p}\mathcal{Y}^*)\left\{\mathbb{E}\left[\mathcal{Y}\mathcal{Y}^*\right]\right\}^{-1}$$

Now we can write this explicitly using our assumptions as,

$$\begin{aligned} \mathbb{E}\mathbf{p}\mathcal{Y}^* &= (\alpha\mathbf{I})\mathcal{X}^*, \\ \mathbb{E}\mathcal{Y}\mathcal{Y}^* &= (\alpha\mathcal{X}\mathcal{X}^* + N_0\mathbf{I}) \end{aligned}$$

Hence we see that the linear estimator using the stochastic criterion (6.32) is given by

$$\mathbf{M} = \alpha\mathcal{X}^*\left[\alpha\mathcal{X}\mathcal{X}^* + N_0\mathbf{I}_{N_T}\right]^{-1} \tag{6.33}$$

Claim 6.3.1.

$$\mathbf{M} = \left[\frac{N_0}{\alpha}\mathbf{I}_{\nu+1} + \mathcal{X}^*\mathcal{X}\right]^{-1}\mathcal{X}^*$$

Proof: Now using (6.33),

$$\mathbf{M} = \mathcal{X}^* \left[\mathcal{X}\mathcal{X}^* + \frac{N_0}{\alpha}\mathbf{I}_{N_T}\right]^{-1}$$

Note that

$$\begin{aligned}
\mathcal{X}^* \left[\mathcal{X}\mathcal{X}^* + \frac{N_0}{\alpha}\mathbf{I}_{N_T}\right] &= \left[\mathcal{X}^*\mathcal{X}\mathcal{X}^* + \frac{N_0}{\alpha}\mathcal{X}^*\right] \\
&= \left[\mathcal{X}^*\mathcal{X} + \frac{N_0}{\alpha}\mathbf{I}_{\nu+1}\right]\mathcal{X}^* \\
\Rightarrow \left[\mathcal{X}\mathcal{X}^* + \frac{N_0}{\alpha}\mathbf{I}_{\nu+1}\right]^{-1}\mathcal{X}^* &= \mathcal{X}^*\left[\mathcal{X}\mathcal{X}^* + \frac{N_0}{\alpha}\mathbf{I}_{N_T}\right]^{-1} \\
&= \mathbf{M}
\end{aligned}$$

□

Hence we can rewrite (6.31) as

$$\hat{\mathbf{p}}_s = \left[\mathcal{X}^*\mathcal{X} + \frac{N_0}{\alpha}\mathbf{I}_{\nu+1}\right]^{-1}\mathcal{X}^*\mathcal{Y}.$$

Comparing this to the solution obtained from the deterministic least squares (6.21), we see that if $\alpha \to \infty$, both are the same! Letting $\alpha \to \infty$ is like saying we are estimating an unknown constant and this is basically what deterministic least squares is trying to do.
Now, let us examine the MMSE of the stochastic estimator, which is given by

$$\mathbb{E}(\mathbf{p} - \hat{\mathbf{p}}_s)(\mathbf{p} - \hat{\mathbf{p}}_s)^* = \alpha\mathbf{I}_{\nu+1} - \alpha\mathcal{X}^*\left(\alpha\mathcal{X}\mathcal{X}^* + N_0\mathbf{I}_{N_T}\right)^{-1}\alpha\mathcal{X}$$

where $\mathbb{E}\mathbf{p}\mathbf{p}^* = \alpha\mathbf{I}$ as before. Using matrix inversion lemma[2] (Lemma 5.7.1) we get,

$$\mathbb{E}(\mathbf{p} - \hat{\mathbf{p}}_s)(\mathbf{p} - \hat{\mathbf{p}}_s)^* = \left[\frac{1}{\alpha}\mathbf{I}_{\nu+1} + \frac{1}{N_0}\mathcal{X}^*\mathcal{X}\right]^{-1}$$

If we make an assumption on $\{x_k\}$ satisfies the training optimality condition of Theorem 6.3.1 (see also (6.30)),

$$\mathcal{X}^*\mathcal{X} = (N_T + \nu)\mathcal{E}_x\mathbf{I}_{\nu+1}$$

then

$$\begin{aligned}
\mathbb{E}(\mathbf{p} - \hat{\mathbf{p}}_s)(\mathbf{p} - \hat{\mathbf{p}}_s)^* &= \left[\frac{1}{\alpha}\mathbf{I}_{\nu+1} + \frac{1}{N_0}(N_T + \nu)\mathcal{E}_x\mathbf{I}_{\nu+1}\right]^{-1} \\
&= \frac{1}{\frac{1}{\alpha} + \frac{(N_T+\nu)\mathcal{E}_x}{N_0}}\mathbf{I}_{\nu+1}
\end{aligned}$$

If

$$(N_T + \nu)\frac{\mathcal{E}_x}{N_0} >> \frac{1}{\alpha},$$

then

$$\mathbb{E}(\mathbf{p} - \hat{\mathbf{p}}_s)(\mathbf{p} - \hat{\mathbf{p}}_s)^* \approx \frac{N_0}{\mathcal{E}_x}\frac{1}{N_T + \nu}\mathbf{I}_{\nu+1} << \alpha\mathbf{I}_{\nu+1}$$

Therefore the error covariance is much smaller than the variance of the random channel, *i.e.,* much smaller than the initial uncertainty.

[2] where the matrix inversion lemma is $[\mathbf{A} + \mathbf{B}\mathbf{C}\mathbf{B}^*]^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}[\mathbf{C}^{-1} + \mathbf{B}^*\mathbf{A}^{-1}\mathbf{B}]^{-1}\mathbf{B}^*\mathbf{A}^{-1}$.

6.4 Problems

Problem 6.1

For a channel with

$$Q(D) = \frac{1}{||p||^2}\left[0.9D + 1.81 + 0.9D^{-1}\right]$$

Assume that the noise variance is $\sigma_n^2 = 0.1$, and $\mathcal{E}_x = 1$.

1. Find G(D) and W(D)for the ZF-DFE.
2. Find SNR_{ZF-DFE}.
3. Design (draw) a Tomlinson precoder.
4. Let $M = 4$ (PAM constellation) for your precoder, Find P_e. Recall that the Tomlinson Precoder output is approximately i.i.d when the input is an i.i.d sequence, and furthermore the output sequence is approximately uniform in distribution over the inteval $[-Md/2, Md/2]$.

Problem 6.2

Consider the channel model in Problem 5.6:

1. Design (draw) a Tomlinson precoder for this channel model and ZF-DFE.
2. Let M be a large value (PAM constellation) for your precoder, Find P_e. Recall that the Tomlinson Precoder output is approximately i.i.d when the input is an i.i.d sequence, and furthermore the output sequence is approximately uniform in distribution over the inteval $[-Md/2, Md/2]$.

Problem 6.3

Consider the $H(D) = 1 + D$ channel. The differential precoder for this channel has the form

$$\bar{m}_k = m_k \ominus \bar{m}_{k-1}$$

where $\ominus$ represents subtraction modulo-M, m_k and $\bar{m}_k$ are the precoder input and output, respectively, and take values in $\{0, \ldots, M-1\}$.

1. Derive the mapping from the precoder output to x_k, the channel input, assuming PAM modulation with symbols separated by distance d.
2. Find the noiseless equivalent channel output. Also determine the input m_k from this output. *Hint*: Apply the inverse of the mapping you found in part (a). You may use the $(\cdot)_M$ operator to show a quantity calculated in M-level arithmetic.
3. Assuming the presence of additive noise, what is the estimate of the input m_k if you apply the same method as in part-2 to determine m_k?

Problem 6.4

[*Tomlinson-Harashima Precoder for the MMSE-DFE*]

1. Derive the expression for $R(D)$, the output of the feedforward filter in the MMSE-DFE.
2. Write an expression for $G_U(D)$, the equvialent unbiased filter for the MMSE-DFE.

3. Using the fact that the $\frac{\text{SNR}_{MMSE-DFE}}{\text{SNR}_{MMSE-DFE,U}}$ scaled value of r_k, the output of the MMSE WMF, is an unbiased MMSE approximation of $X(D)G_U(D)$, find the feedback filter, $B(D)$ for the Tomlison-Harashima precoder for the MMSE-DFE. Hence find $r_{U,k}$, the scaled output of the feedforward filter.

4. Determine the output of the precoder after the modulo operation, $\Gamma(r_{U,k})$.

Problem 6.5

1. Suppose we are given a periodic channel $h(t) = h(t+T)$. Show that $\varphi_n(t) = e^{j\frac{2\pi}{T}nt}$ are eigenfunctions of the channel. (Hint: express $h(t)$ in a Fourier series). By eigenfunctions, we mean that if $\phi(t)$, $t \in [0,T)$ is used for a periodic channel $h(t)$, the output $y(t)$, $t \in [0,T)$, will be a scaled version of $\phi(t)$. Note here, that both the input and the output signals are restricted to the interval $[0,T)$.

2. What are the associated eigenvalues ?

3. If the channel is not periodic, but had finite length ν, how could the designer create a new set of functions $\widehat{\varphi}_n(t)$ from $\varphi(t)$ such that the output looks as if it was created by a periodic channel.

4. If the receiver only looks at the periodic part of the output signal, what would the associated eigenvalues be ?

5. If a transmission uses the channel eigenfunctions developed here, how much faster does the sampling rate hav to be ? (or how much more bandwith does the design have to use ?)

Problem 6.6

Assume an OFDM system with N subcarriers, of symbol length T, of which ν is the length of the cyclic prefix

$$\phi_k(t) = \begin{cases} \frac{1}{\sqrt{T-\nu}} e^{j2\pi\frac{W}{N}k(t-\nu)} & , \quad \text{if}\, t \in [0,T] \\ 0 & , \quad \text{otherwise} \end{cases}$$

On a particular symbol, we have the transmitted signal given by:

$$s_l(t) = \sum_{k=0}^{N-1} x_{k,l}\phi_k(t-lT)$$

Then the output from a transmitter is:

$$s(t) = \sum_{l=0}^{\infty} s_l(t)$$

Moreover we assume that the impulse response $g(t)$ of the channel is of length ν.

1. Compute the received signal $r(t)$. (consider an AWGN channel noise $n(t)$)

2. The OFDM receiver consists of a filter bank, matched to the last part $[\nu, T]$ of the transmitter waveforms $\phi_k(t)$:

$$\psi_k(t) = \begin{cases} \phi_k^*(T-t) & , \quad \text{if}\, t \in [0, T-\nu] \\ 0 & , \quad \text{otherwise} \end{cases}$$

Compute the sampled output from the receiver filter banks $y_k = (r * \psi_k)(t)\mid_{t=T}$ and show that it is of the form $y_k = h_k x_k + n_k$. (Remember that since the cyclic prefix contains all ISI from previous symbol, we can ignore the time index l)

Problem 6.7

Given an OFDM system with N subcarriers , compute the channel estimation in the frequency domain. Do we need N pilot symbols?

Problem 6.8

Given an OFDM system with $N = 4$ sub-carriers, cyclic prefix and a given channel $P(D) = 1 + 1.81D + 0.81D^2$:

1. Find the matrix $\mathbf{P}$ and compute its eigen decomposition.
2. After applying FEQ, compute the SNR on each sub carrier.

Assume that $\mathcal{E}_x = 0.1$ and $\sigma^2 = 0.01$.

Problem 6.9

Consider a finite impulse response channel

$$\mathbf{y}_k = \sum_{n=0}^{\nu} \mathbf{p}_n x_{k-n} + \mathbf{z}_k$$

where $\mathbf{y}_k,\ \mathbf{z}_k \in \mathbb{C}^2$ and $\mathbf{p}_n \in \mathbb{C}^2$, i.e they are 2-dimensional vectors. This could arise, for example, through Nyquist sampling like the model considered in class ($L = 2$).

1. Suppose one observes a block of N samples of $\{\mathbf{y}_k\}$, $\mathcal{Y}_k = \begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix}$. Write down the relationship between $\mathcal{Y}_k$ and $\mathcal{X}_k = \begin{bmatrix} x_k \\ \vdots \\ x_{k-N+1} \\ \vdots \\ x_{k-N+1-\nu} \end{bmatrix}$ in the form

$$\mathcal{Y}_k = \mathbf{P}\mathcal{X}_k + \mathcal{Z}_k$$

   where $\mathcal{Y}_k,\ \mathcal{Z}_k \in \mathbb{C}^{2N}$, $\mathcal{X}_k \in \mathbb{C}^{N+\nu}$, $\mathbf{P} \in \mathbb{C}^{2N\times(N+\nu)}$ by specifying the form of $\mathbf{P}$.

2. Suppose we use a cyclic prefix, i.e

$$x_{k-N-l} = x_{k-l},\ \ l = 0, \cdots, \nu - 1$$

   Develop the equivalent model:

$$\mathcal{Y}_k = \widetilde{\mathbf{P}}\widetilde{\mathcal{X}}_k + \mathcal{Z}_k \tag{6.34}$$

   where $\widetilde{\mathcal{X}}_k = \begin{bmatrix} x_k \\ \vdots \\ x_{k-N+1} \end{bmatrix} \in \mathbb{C}^N$ and $\widetilde{\mathbf{P}} \in \mathbb{C}^{2N\times N}$. Find $\widetilde{\mathbf{P}}$.

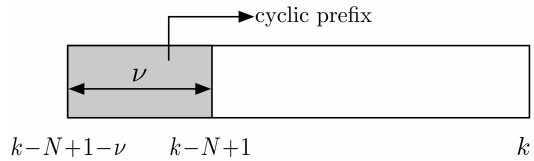


Figure 6.16: Cyclic Prefix

3. Let $\mathbf{Y}(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{y}_k e^{-j\frac{2\pi}{N}k\ell}$, $\mathbf{Z}(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{z}_k e^{-j\frac{2\pi}{N}k\ell}$, $\mathbf{P}(\ell) = \sum_{n=0}^{\nu} \mathbf{p}_n e^{-j\frac{2\pi}{N}n\ell}$, $X(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}k\ell}$.

   Develop the vector OFDM form for (6.34), i.e, show that

   $$\mathbf{Y}(\ell) = \mathbf{P}(\ell)X(\ell) + \mathbf{Z}(\ell), \quad \ell = 0, \cdots, N-1 \tag{6.35}$$

   This can be done by either arguing about equivalent periodic sequences or any other proof technique. Here we would like to see a derivation, just stating the result is not enough.

4. In the form given in (6.35), we get N parallel vector channels. If we want to detect each component $\{X(\ell)\}$ separately what would be the best linear estimator of $X(\ell)$ from $\mathbf{Y}(\ell)$, i.e the appropriate "frequency-domain" MMSE linear equalizer (FEQ).

Problem 6.10

Find the MMSE-LE filter that normalizes the following channel

$$P(D) = 0.7 + 1.5D + 0.8D^2$$

to a target channel of $\tilde{H}(D) = 1 + D$. In this case our predicition error becomes

$$E(D) = X(D)\tilde{H}(D) - W(D)Y(D).$$

You may assume that $R_{xx} = \mathcal{E}_x$ and that there is additive white Gaussian noise of power N_0.
The goal of this exercise to show the links between equalization, and the other methods of dealing with ISI that we looked at such as OFDM and sequence detection. In pratical systems a channel is rarely fully equalized to a delta function ($\tilde{H}(D) = 1$). Instead the channel is equalized to something relatively benign (such as $1 + D$), and then OFDM or sequence detection is applied to this equivalent channel.
We see that the techniques we have learned for dealing with ISI channels do not have to be used independently, and indeed in practical systems equalization is often combined with another method for coping with ISI.

Problem 6.11

Given an OFDM system with $N = 8$ subcarriers, and a given channel $P(D) = 1 + 0.9D$:

1. Find the matrix P and compute its eigen decomposition.
2. After applying FEQ, compute the SNR on each subcarrier.
3. Calculate the error rate for QPSK modulation.
4. What is the worst error rate?

Problem 6.12

[*Sensitivity of OFDM to frequency offset*]
To the $P(D) = 1 + D$ channel, add a small frequency offset, $\delta\omega$. This will make the $\tilde{\mathbf{P}}$ matrix not exactly circulant anymore. Calculate the inter-carrier interference and quantify for what frequency offset it becomes significant. Let $N = 4$. $\delta\omega = \omega_o T$, where ω_o is the frequency offset and T is the symbol period. $p(n)$ is replace to include the frequency offset such that

$$y(k) = \sum_{n=0}^{\nu} e^{j\delta\omega k} p(n)x(k-n) + z(k)$$

Hint: Calculate the new $\tilde{\mathbf{P}}$ matrix, then calculate the Λ matrix.

Problem 6.13

[Inter-carrier Interference in OFDM]
Consider the scalar discrete-time inter symbol interference channel considered in the class,

$$y_k = \sum_{n=0}^{\nu} p_n x_{k-n} + z_k, \quad k = 0, \ldots, N-1, \tag{6.36}$$

where $z_k \sim \mathbf{C}\mathcal{N}(0, \sigma_z^2)$ and is i.i.d., independent of $\{x_k\}$. Let us employ a cyclic prefix as done in OFDM, *i.e.*,

$$x_{-l} = x_{N-1-l}, \quad l = 0, \ldots, \nu.$$

As done in class given the cyclic prefix,

$$\mathbf{y} = \begin{bmatrix} y_{N-1} \\ \vdots \\ y_0 \end{bmatrix} = \underbrace{\begin{bmatrix} p_0 & \ldots & \ldots & p_\nu & 0 & \ldots & 0 & 0 \\ 0 & p_0 & \ldots & p_{\nu-1} & p_\nu & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ldots & \ldots & 0 & p_0 & \ldots & \ldots & p_\nu \\ p_\nu & 0 & \ldots & 0 & 0 & p_0 & \ldots & p_{\nu-1} \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ p_1 & \ldots & p_\nu & 0 & \ldots & 0 & 0 & p_0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}}. \tag{6.37}$$

In the derivation of OFDM we used the property that

$$\mathbf{P} = \mathbf{F}^* \mathbf{D} \mathbf{F}, \tag{6.38}$$

where

$$\mathbf{F}_{p,q} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N}(p-1)(q-1)\right)$$

and $\mathbf{D}$ is the diagonal matrix with

$$\mathbf{D}_{l,l} = d_l = \sum_{n=0}^{\nu} p_n e^{-j\frac{2\pi}{N}nl}.$$

Using this we obtained

$$\mathbf{Y} = \mathbf{F}\mathbf{y} = \mathbf{D}\mathbf{X} + \mathbf{Z},$$

where $\mathbf{X} = \mathbf{F}\mathbf{x}$, $\mathbf{Z} = \mathbf{F}\mathbf{z}$. This yields the parallel channel result

$$\mathbf{Y}_l = d_l \mathbf{X}_l + \mathbf{Z}_l. \tag{6.39}$$

If the carrier synchronization is not accurate, then (6.36) gets modified as

$$y(k) = \sum_{n=0}^{\nu} e^{j2\pi f_0 k} p_n x_{k-n} + z_k, \quad k = 0, \ldots, N-1 \tag{6.40}$$

where f_0 is the carrier frequency offset. If we still use the cyclic prefix for transmission, then (6.37) gets modified as

$$\underbrace{\begin{bmatrix} y(N-1) \\ \cdot \\ \cdot \\ y(0) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} p_0 e^{j2\pi f_0 (N-1)} & \ldots & p_\nu e^{j2\pi f_0 (N-1)} & 0 & \ldots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \ddots & \ddots & \ddots & e^{j2\pi f_0 \nu} p_0 & \ldots & e^{j2\pi f_0 \nu} p_\nu \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ e^{j2\pi f_0 0} p_1 & \ldots & e^{j2\pi f_0 0} p_\nu & 0 & \ldots & 0 & e^{j2\pi f_0 0} p_0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}}. \tag{6.41}$$

i.e.,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$

Note that

$$\mathbf{H} = \mathbf{S}\mathbf{P},$$

where $\mathbf{S}$ is a diagonal matrix with $\mathbf{S}_{l,l} = e^{j2\pi f_0 (N-l)}$ and $\mathbf{P}$ is defined as in (6.37).

(a) Show that for $\mathbf{Y} = \mathbf{F}\mathbf{y}$, $\mathbf{X} = \mathbf{F}\mathbf{x}$,

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \mathbf{Z} \tag{6.42}$$

and prove that

$$\mathbf{G} = \mathbf{F}\mathbf{S}\mathbf{F}^* \mathbf{D}.$$

(b) If $f_0 \neq 0$, we see from part (a) that $\mathbf{G}$ is no longer a diagonal matrix and therefore we do not obtain the parallel channel result of (6.39). We get inter-carrier interference (ICI), *i.e.,* we have

$$\mathbf{Y}_l = \mathbf{G}_{l,l} \mathbf{X}_l + \underbrace{\sum_{q \neq l} \mathbf{G}(l,q) \mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0, \ldots, N-1,$$

which shows that the other carriers interfere with $\mathbf{X}_l$. Compute the SINR (signal-to-interference plus noise ratio). Assume $\{\mathbf{X}_l\}$ are i.i.d, with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$. You can compute the SINR for the particular l and leave the expression in terms of $\{G(l,q)\}$.

(c) Find the filter $\mathbf{W}_l$, such that the MMSE criterion is fulfilled,

$$\min_{\mathbf{W}_l} \mathbb{E}|\mathbf{W}_l^* \mathbf{Y} - \mathbf{X}_l|^2.$$

You can again assume that $\{\mathbf{X}_l\}$ are i.i.d with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$ and that the receiver knows $\mathbf{G}$. You can now state the answer in terms of $\mathbf{G}$.

(d) Find an expression for $\mathbf{G}_{l,q}$ in terms of $f_0, N, \{d_l\}$. Given this and (b), what can you conclude about the value of f_0. For what values of f_0 do you think that the inter-carrier interference problem is important? *Hint:* Use the summation of the geometric series hint given in first page.

Problem 6.14

Consider the channel estimation problem based on the deterministic criterion. Let the channel be:

$$\mathcal{Y} = \mathcal{X}\mathbf{p} + \mathbf{z},$$

where channel memory length $\nu = 1$ and length of the training sequence $N_T = 2$. The deterministic criterion to estimate $\mathbf{p}$ is given by:

$$\hat{\mathbf{p}}_{\text{opt}} = \text{argmin}_{\mathbf{p}} ||\mathcal{Y} - \mathcal{X}\mathbf{p}||.$$

1. Assume that all the signal are real, prove directly by differentiating that the optimum $\mathbf{p}$ satisfies

$$\hat{\mathbf{p}}_{\text{opt}} = (\mathcal{X}^*\mathcal{X})^{-1}\mathcal{X}^*\mathcal{Y}. \tag{6.43}$$

   Prove eqn(6.43) when the signals can be complex.

2. Let $\mathcal{E}_x = 0.1$, $\sigma^2 = 0.01$. Find the expected error in the estimation assuming that the training sequence is the optimal sequence obtained under the constraint $\text{Tr}(\mathcal{X}^*\mathcal{X}) \leq (\nu + 1)(N_T + \nu)\mathcal{E}_x$.

Part III

Wireless Communications

Chapter 7

Wireless channel models

Wireless communication refers to communication using electromagnetic waves propagated in "air" to carry information. This is different from the telephone line channel or the cable channel that required a wire to carry information across from point to point. Today wireless networks have significantly penetrated our lives through cellular phones, satellite TV and even cordless phones which combine with the traditional PSTN wired telephone line network.

Wireless communication has a long history, dating back to 1897 when Marconi successfully demonstrated wireless telegraphy. The early 20th century witnessed the birth of radio transmission with speech and audio carried across the ocean. Soon, television transmission started in the 1930s and the launching of satellites in the 1950s started long-haul wireless networks. Today microwave networks for line-of-sight transmission, cellular networks for wireless telephony and wireless LANs for data access seem ubiquitous. In this part of the class we introduce the transmission medium in cellular wireless communication and some of the signal processing issues dealing with reception of multiple user wireless transmissions.

There are two main distinguishing features of wireless communications in comparison to wired channels. The first is the channel could be time-varying causing the received signal strength to wax and wane. This phenomenon is called *channel fading*, and occurs due to radio wave propagation effects. The second is that the wireless channel is an inherently shared medium and hence multiple user interact and interfere with each other. Therefore we need to devise techniques to handle multiuser interference making it distinct from the point-to-point (single-user) detectors we studied in Chapter 2. In general we also encounter inter-symbol interference in wireless channels, but this is something we have already encountered in Chapter 4-6.

A cellular network consists of users spread over a geographical area served by a "base station". The area of a single base station is often called a "cell" and hence the name cellular communication. Several base-stations combine to cover large geographical areas. For example New York city has base-stations every couple of blocks and therefore the cells could be quite small. On the other hand in sub-urban and rural areas the cells could be quite large, of the order several square kilometers.

The base-stations connect to the wired telephone network enabling calls to and from the telephone line network to the wireless cellular network. Given this background, we study the following aspects of wireless networks.

- Radio wave propagation in scattering environments.
- Physical and statistical models for wireless channels.
- Detection in wireless point-to-point (single users) channels.
- Concept of diversity in various forms.

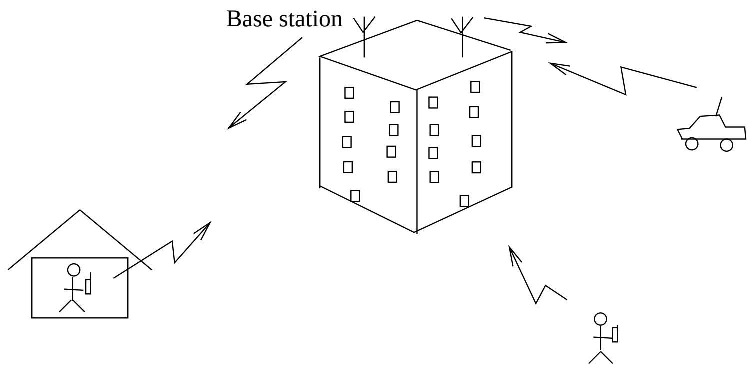


Figure 7.1: A cartoon of a cellular base station.

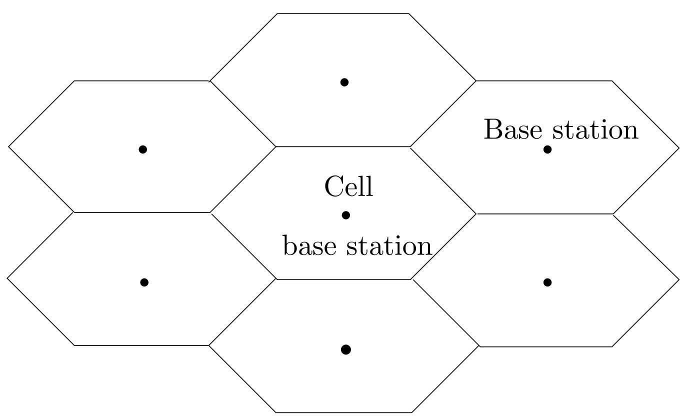


Figure 7.2: An oversimplified view of cells in geographical area.

- Wireless multi-users channels, in particular focus on the multiple access channel.
- Multiuser detection for multiple access channels.

In this chapter we study wireless channel models after understanding the effects of radio wave propagation. In Chapter 8, we study detection in point-to-point (single user) wireless channels. In Chapter 9, we study multiuser wireless channels and focus on multiple access channels.

7.1 Radio wave propagation

Wireless transmission occurs by modulating electromagnetic waves which are then transmitted through an antenna. The receiver processes the impinging electromagnetic wave and detects the transmitted information. The typical carrier frequencies are 0.9 GHz, 1.8 GHz, 2.4 GHz, etc. for cellular wireless networks. The main effects on the information transmission arises due to the propagation effects on the electromagnetic waves due to reflection, absorption and constructive and destructive combining of these electromagnetic waves. Therefore we first study these effects and their impact on communication.

7.1.1 Free space propagation

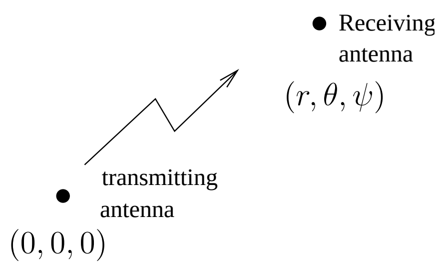


Figure 7.3: Free space radio propagation.

Free space propagation occurs when the transmitter and the receiver have an unobstructed line-of-sight path between them. For a *fixed* transmitting and *fixed* receiving antenna, and for a pure sinusoid $\cos 2\pi ft$ transmitted, the observed electric field (far field) at time t and position (r,θ,ψ) is

$$\mathbf{E}[f,t,(r,\theta,\psi)] = \frac{1}{r}\mathrm{Re}[\alpha_s(\theta,\psi,f)e^{j2\pi f(t-r/c)}] \tag{7.1}$$

where $\alpha_s(\theta,\psi,f)$ is the radiation pattern of transmitting antenna, $c = 3\times 10^8$ (velocity of light). Note that phase variation of fr/c is just due to the delay caused by radiation travelling at the speed of light. The receiving antenna could have a radiation (reception) pattern of its own and this could modify the observed electric field. This just means $\alpha(\theta,\psi,f)$ replaces $\alpha_s(\theta,\psi,f)$ where $\alpha(\theta,\psi,f) = \alpha_s(\theta,\psi,f)\alpha_r(\theta,\psi,f)$. Therefore the received electric field at the antenna is

$$\mathbb{E}[f,t,(r,\theta,\psi)] = \frac{1}{r}\mathrm{Re}[\alpha(\theta,\psi,f)e^{j2\pi f(t-\frac{r}{c})}]. \tag{7.2}$$

Notes:

1. Notice that in (7.2) the electric field decays as $\frac{1}{r}$ as a function of distance between transmitting and receiving antennas. Thus the power behaves as,

$$P_r = \frac{\mid \alpha(\theta,\psi,f) \mid^2}{r^2} \tag{7.3}$$

and hence decays as $\frac{1}{r^2}$, as a function of the distance. This is often called the squared law loss and determines how far we can have the transmitter and receiver, implying the coverage area of a base-station for example.

2. The relationship in (7.2) holds only for free space propagation, *i.e.* no absorption, no reflections or scattering.

3. The radiation pattern of the transmitting antenna could be directional and also cause phase variations.

4. It is clear that (7.2) does *not* hold for $r = 0$, *i.e.,* this is a *far field* model when the transmitter and the receiver are far apart.

5. **Linearity:** The observed electric field is linear in the input, *i.e.* if we transmit a sum (superposition) of two sinusoids, the output will be the sum of the outputs one would have observed if each of them were transmitted alone. This is an important feature of the propagation behavior that will be used in the channel models later.

7.1.2 Ground Reflection

In typical scenarios, an unobstructed line-of-sight path between the transmitter and the receiver is rare. We will now look at our first modification of the free space model developed in Section 7.1.1.
A two-ray ground reflection model is a usual first step in demonstrating the kind of impairments we would see in a real propagation environment. Even though such a model (shown in Figure 7.4) does not capture all the complexities of a real-world channel, it gives us a taste of what to expect.
In Figure 7.4, we consider a typical scenario where one antenna (say the transmitter) is much higher (at a height h_t) than the receiver (at height h_r). This might represent a base-station transmitting to a mobile unit. We consider a perfect omnidirectional transmitter and receiver, *i.e.,* $\alpha(\theta, \psi, f) = 1$.

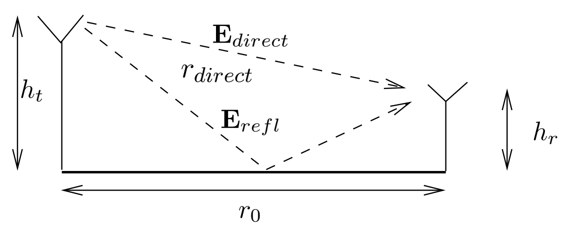


Figure 7.4: A two-ray ground reflection model.

We assume that receiver observes two versions of the transmitted wave, one which reaches it directly and we call this E_{direct} and the other which reflects off the ground and reaches it after travelling a total distance of r_b. Due to linearity, the total electric field is the sum of these two components given by,

$$\mathbf{E}(f,t) = \mathbf{E}_{direct}(f,t) + \mathbf{E}_{refl}(f,t) \tag{7.4}$$

Let E_0 be the observed electric field at a reference distance d_0 from the transmitter. If the ground is a perfect reflector, then

$$\begin{aligned}
\mathbf{E}_{direct} &= \frac{E_0 d_0}{r_d} \cos\left(2\pi f\left(t - \frac{r_d}{c}\right)\right) \\
\mathbf{E}_{refl} &= \frac{-E_0 d_0}{r_b} \cos\left(2\pi f\left(t - \frac{r_b}{c}\right)\right)
\end{aligned}$$

where r_d is the distance travelled by the direct wave, *i.e.*, r_d is the distance from transmitter to receiver and r_b is the total distance travelled by reflected wave.
Now, let us define the path difference Δ as,

$$\begin{aligned}
\Delta &= r_b - r_d \overset{(a)}{=} \sqrt{(h_t+h_r)^2+r_0^2} - \sqrt{(h_t-h_r)^2+r_0^2} \\
&= r_0\left[1+\left(\frac{h_t+h_r}{r_0}\right)^2\right]^{1/2} - r_0\left[1+\left(\frac{h_t-h_r}{r_0}\right)^2\right]^{1/2} \\
&\overset{(b)}{\approx} r_0\left[1+\frac{1}{2}\left(\frac{h_t+h_r}{r_0}\right)^2 - 1 - \frac{1}{2}\left(\frac{h_t-h_r}{r_0}\right)^2\right] \\
&= \frac{1}{2r_0}\left[(h_t+h_r)^2-(h_t-h_r)^2\right] = \frac{1}{2r_0}(4h_th_r) = \frac{2}{r_0}h_th_r
\end{aligned}$$

where (a) occurs due to geometry in Figure 7.5 *i.e.*,

$$\begin{aligned}
r_b^2 &= (h_t+h_r)^2+r_0^2 \\
r_d^2 &= (h_t-h_r)^2+r_0^2
\end{aligned} \tag{7.5}$$

and the approximation (b) is for $r_0 >> (h_t+h_r)$. This means that the distance between the transmitter and the receiver are much larger than the antenna heights. Hence we get

$$\Delta = r_b - r_d \approx \frac{2}{r_0}h_th_r \tag{7.6}$$

Therefore using this in (7.4) we get

$$\begin{aligned}
\mathbf{E}(f,t) &= \frac{E_0d_0}{r_d}\cos\left[2\pi f\left(t-\frac{r_d}{c}\right)\right] - \frac{E_0d_0}{r_b}\cos\left[2\pi(ft-f\frac{r_b}{c})\right] \\
&\approx \frac{E_0d_0}{r_0}\cos\left[2\pi f\left(t-\frac{r_d}{c}\right)\right] - \frac{E_0d_0}{r_0}\cos\left[2\pi f(t-\frac{r_b}{c})\right]
\end{aligned}$$

where the approximation is because $r_b = \sqrt{(h_t+h_r)^2+r_0^2} \approx r_0$ and $r_d = \sqrt{(h_t+h_r)^2+r_0^2} \approx r_0$ in the first order.

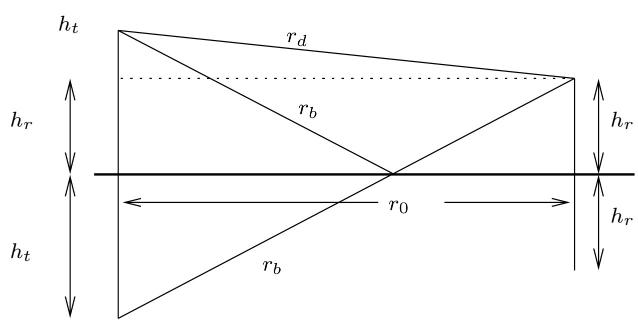


Figure 7.5: Trigonometry to find path differences.

Now, let us consider the complex representation of the electric field. One can get to the previous representation by taking the real part.

$$\begin{aligned}\mathbf{E}(f,t) &= \frac{E_0 d_0}{r_d} e^{j\omega\left(t-\frac{r_d}{c}\right)} - \frac{E_0 d_0}{r_b} e^{j\omega\left(t-\frac{r_b}{c}\right)} \\ &\approx \frac{E_0 d_0}{r_0}\left[e^{j\omega\left(t-\frac{r_d}{c}\right)} - e^{j\omega\left(t-\frac{r_b}{c}\right)}\right]\end{aligned}$$

where as before the approximation is that $\frac{1}{r_d} \approx \frac{1}{r_b} \approx \frac{1}{r_0}$ in the far field. Moreover we have used the notation $\omega = 2\pi f$, for brevity.
Using this we get

$$\begin{aligned}\mathbf{E}(f,t) &\approx \frac{E_0 d_0}{r_0} e^{j\omega t}\left[e^{-j\omega\frac{r_d}{c}} - e^{-j\omega\frac{r_b}{c}}\right] \\ &= \left[\frac{E_0 d_0}{r_0}\right] e^{j\omega t} e^{-j\omega\frac{r_b}{c}}\left[e^{j\omega\left(\frac{r_b-r_d}{c}\right)} - 1\right].\end{aligned}$$

Now to find the power we need the total magnitude of the electric field, *i.e.*,

$$|\mathbf{E}(f,t)| = \left|\frac{E_0 d_0}{r_0}\right|\left|e^{j\omega\left(\frac{r_b-r_d}{c}\right)} - 1\right|$$

Consider the second term above,

$$\begin{aligned}\left|e^{j\omega\left(\frac{r_b-r_d}{c}\right)} - 1\right|^2 &= \left\{\cos\left[\omega\left(\frac{r_b-r_d}{c}\right)\right] - 1\right\}^2 + \sin^2\left[\omega\left(\frac{r_b-r_d}{c}\right)\right] \\ &= 1 + 1 - 2\cos\left[\omega\left(\frac{r_b-r_d}{c}\right)\right] \\ &= 2\left\{1 - \cos\left[\omega\left(\frac{r_b-r_d}{c}\right)\right]\right\} \\ &= 2\left\{2\sin^2\left[\omega\left(\frac{r_b-r_d}{2c}\right)\right]\right\}\end{aligned}$$

Therefore now we use the approximation developed in (7.6) as,

$$\begin{aligned}|\mathbf{E}(f,t)| &\approx \left|\frac{E_0 d_0}{r_0}\right| 2\sin\left[\omega\left(\frac{r_b-r_d}{2c}\right)\right] \\ &\overset{(a)}{\approx} \left|\frac{E_0 d_0}{r_0}\right| 2\sin\left(\omega\frac{2h_t h_r}{r_0 2c}\right) \\ &\overset{(b)}{\approx} \left|\frac{E_0 d_0}{r_0}\right| \frac{2\omega}{c}\frac{h_t h_r}{r_0} \\ &= \frac{1}{r_0^2}\left[\frac{2E_0 d_0 h_t h_r}{c}\right]\end{aligned}$$

where (a) occurs due to the approximations in (7.6) and (b) occurs because $\sin\theta \approx \theta$ for small θ.
Hence the power received is $\mid E(f,t) \mid^2$ is $P_r \approx \frac{1}{r_0^4}\left[\frac{4E_0 d_0 h_t h_r}{c}\right]$.
Hence, the received power decays as $\frac{1}{r_0^4}$ instead of $\frac{1}{r_0^2}$ as was seen in the free space propagation model. This drastically different behavior in the power decay shows that complex reflections can alter the decay behavior with distance.
The main ideas from radio wave propagation considered in Section 7.1.1 and 7.1.2 can be summarized as follows.

1. The free space transmission resulted in a linear time-invariant system with a power decay of $\frac{1}{r_0^2}$, where r_0 was the distance between sender and receiver.

2. For a ground reflection, which occurs frequently in rural and suburban environments, the power loss become $1/r_0^4$ *i.e.,* it decays much faster than in free space. Therefore in general the radio wave propagation causes a decay of power as

$$P_r \propto \frac{1}{r^\beta} \tag{7.7}$$

where $\beta \geq 2$. Typically $\beta \in [2, 6]$.

7.1.3 Log-normal Shadowing

The model given in (7.7) may not completely capture the power loss behavior. Unfortunately the radio propagation environment is much more complicated than just a reflection model. Typically radio waves get reflected, absorbed, scattered and diffracted. All these depend very highly on the particular propagation environment. For example, the presence of a large building causing absorption, or a rough surface causing scattering means that one cannot easily predict the radio wave propagation behavior. Here is where we first encounter a *stochastic model* for the propagation, where the randomness corresponds to the unknown or uncertain propagation environment.

Given that just the "distance loss" behavior of (7.7) does not account for the complete effects of radio propagation, we introduce the notion of *shadowing* to account for the randomness in the propagation environment. It has been empirically observed that the received signal power $P_r(r_0)$ as a function of distance r_0 between transmitter and receiver behaves as,

$$P_r(r_0) \propto \frac{\mathcal{S}}{(r_0)^\beta} \tag{7.8}$$

where $\mathcal{S}$ is a random variable (shadowing) accounting for some of the uncertainties in the propagation environment. Through empirical studies it has been observed that the random variable $\mathcal{S}$ behaves as a log-normal distribution, *i.e.,* $\log \mathcal{S}$ is a zero-mean Gaussian distributed random variable with a given variance. This model attempts to account for the variability in the empirical signal seen, even for the same distance separation between the transmitter and the receiver. Therefore the model in (7.8) shows that the received signal levels at a particular transmitter-receiver separation of r_0 is a random variable. In terms of signal power measured in dB,

$$10 \log P_r(r_0) = \mathcal{K} + 10 \log \mathcal{S} - 10\beta \log r_0,$$

where $\mathcal{K}$ is the proportionality constant. Since $\log \mathcal{S}$ is modelled as zero-mean, the mean received power in dB is that given by the distance loss and the shadowing adds on to this with a normal distribution.

In summary the shadowing represents the absorption and clutter loss which depends on the unknown (and uncertain) propagation environment. This is our first encounter with such randomness in the modelling. We will encounter it again in the next section where we consider mobility. Note, however that the shadowing loss is modelled only for fixed environments. The modelling of shadowing for mobile environments is quite difficult.

7.1.4 Mobility and multipath fading

Up to now the propagation effects have been perhaps random but time-invariant. In this section we introduce one of the distinguishing features of wireless channels, *i.e., time-variation.* These rapid fluctuations in signal strength are called fading and are caused by constructive and destructive interference of reflected waves. We have already seen the effect of such superposition in Section 7.1.2 where we saw the combination of a direct and a ground reflected wave. The essential difference here is that the receiver and/or the transmitter is mobile and hence these superpositions change over time causing the rapid signal strength fluctuations. These are *small-scale* effects in that the distance travelled is of the order of several wavelengths of the carrier wave in order to observe these variations. Therefore the signal strength variations *do not* occur due to changes in the propagation environment (*i.e.,* reflectors, etc.) but due to phase changes in the "multipath" superposed reflections themselves.

Doppler Shift: Consider a mobile moving at a constant velocity v, along a path AB as illustrated in Figure 7.6. Suppose, as shown in Figure 7.6 there is a remote fixed transmitter $\mathcal{T}$.

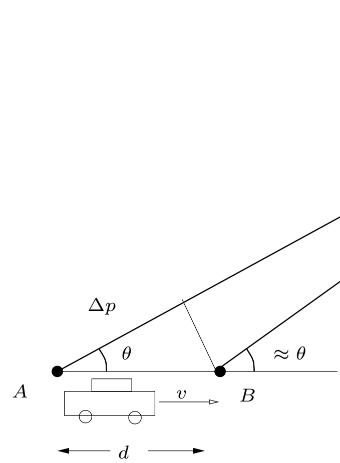


Figure 7.6: Doppler shift.

The difference in path length travelled by a wave from $\mathcal{T}$ to the mobile at A and B is $\Delta p = d\cos\theta = v\Delta t\cos\theta$ where Δt is the time needed for the car to go from A to B and θ is the angle between AB and $A\mathcal{T}$. Here we have assumed that $\mathcal{T}$ is far enough that the angles from A and B are the same, *i.e.,* θ. Therefore due to the path length difference the phase difference of the radio wave is

$$\Delta\phi = \frac{2\pi\Delta p}{\lambda} = \frac{2\pi v\Delta t}{\lambda}\cos\theta$$

where λ is the wavelength of the carrier. Therefore the apparent change in frequency is given by

$$f_d = \frac{1}{2\pi}\frac{\Delta\phi}{\Delta t} = \frac{v}{\lambda}\cos\theta. \tag{7.9}$$

The relationship given in (7.9) relates the mobile speed to the change in frequency, which is called the *Doppler shift*. This is a phenomenon one might have encountered often when a police siren seems to increase in frequency while approaching while decreasing when it is receding. Though the siren is always transmitting at the same frequency a fixed listener observes a variation in the frequency and this effect is called the Doppler effect after its discoverer. Clearly, in (7.9) we see that if the mobile is moving towards $\mathcal{T}$, *i.e.,* $\theta > 0$, then the frequency shift is positive (*i.e.,* apparent frequency is higher) and the shift is negative when it is receding from $\mathcal{T}$ (*i.e.,* $\theta < 0$).

Now, this Doppler shift is the cause for the time-variation seen in wireless channels. The situation becomes more complex when the mobile receives not one copy but several reflected copies of the signal transmitted by $\mathcal{T}$. This can be modelled as having several virtual sources $\{\mathcal{T}_l\}$ transmitted to the mobile.

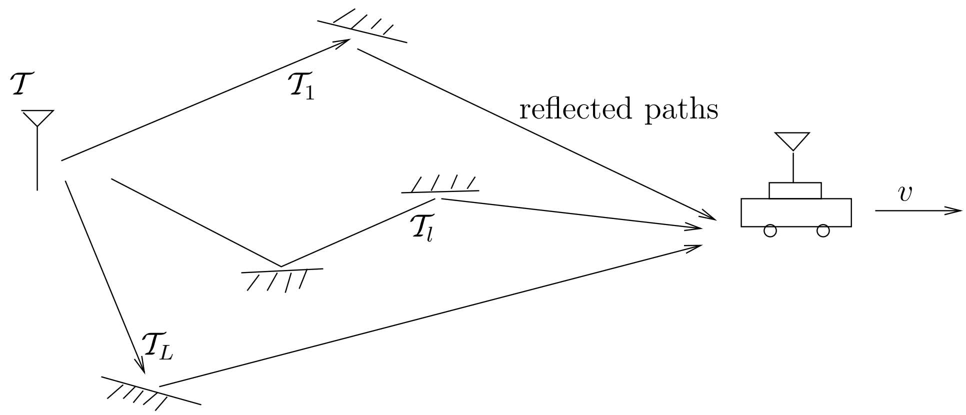


Figure 7.7: Multipath reflections and Doppler shifts.

Now, consider the effect of (7.9) on the received electric field, given in (7.1). The change in frequency causes the received field at point B to be

$$E(f,t) \approx \frac{1}{r_0}\mathrm{Re}[\alpha(\theta,\psi,f)e^{j2\pi(f+f_d)t}]$$

For an ideal omni-directional transmitter and receiver, the complex electric field is given by

$$E(f,t) = \left[\frac{1}{r_0}e^{j2\pi ft}\right]e^{j2\pi f_d t}$$

Therefore in the baseband and after compensation of distance loss we get

$$E_b(t) = e^{j2\pi f_d t}$$

Now, this is for a single path and for reflections we get several virtual sources, *i.e.,* the received baseband signal is

$$y(t) = \sum_{l=1}^{L} c_l e^{j2\pi\left[\frac{v}{\lambda}\cos\theta_l\right]t} \tag{7.10}$$

where $\{\theta_l\}$ are the angles to the different reflectors and $\{c_l\}$ are the (complex) reflection coefficients of each of the reflectors. This is because each of the reflectors could cause phase shift as well as absorb to reduce the amplitude. This is illustrated in Figure 7.7.

In general, the reflectors could be such that one also gets delayed versions of the signal, but this is an effect we will study in Section 7.2.

The model given in (7.10) was for a single sinusoid sent at a carrier frequency and it showed the baseband equivalent signal. Now, if a narrowband signal $x(t)$ was sent instead of a pure sinusoid, then the received signal would be,

$$\begin{aligned} y(t) &= \left[\sum_{l=1}^{L} c_l e^{j2\pi\frac{v}{\lambda}\cos\theta_l t}\right]x(t) \\ &\triangleq f(t)x(t) \end{aligned} \tag{7.11}$$

The relationship in (7.11) illustrates two things. One, that the equivalent channel is multiplicative, *i.e.,* the channel $f(t)$ is time-varying and causes the received signal strength fluctuations. Second, the overall system behavior is that of a *linear time-varying* system.

7.1.5 Summary of radio propagation effects

There were three main effects that we studied in Sections 7.1.1-7.1.4, *i.e.,* distance loss effect, shadowing and multipath fading. Overall, for a baseband signal if a narrowband signal $x(t)$ is transmitted, then the received signal is

$$y(t) = \frac{\mathcal{S}}{(r_0)^{\frac{\beta}{2}}} f(t)x(t),$$

where $\mathcal{S}$ represents the random log-normal shadowing described in Section 7.1.3, $(r_0)^{\frac{-\beta}{2}}$ loss describes the distance loss and $f(t)$ describes the small-scale fast fading alternation. In most baseband systems, the relatively constant qualities of the shadowing and distance loss one compensated by automatic gain control. Therefore the multipath fading becomes the dominant effect for a communication system design and analysis. Clearly the overall model depends on unknown propagation environment that changes over time. Thus leads to modelling the channel as a random variable with given probability laws which are characterized empirically.

7.2 Wireless communication channel

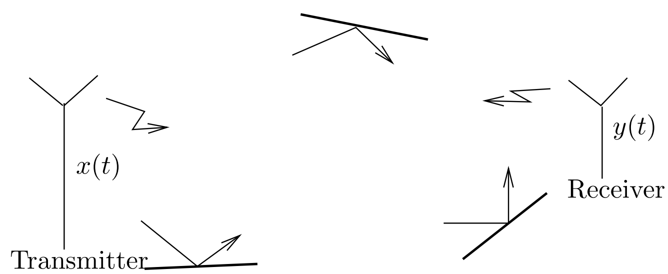


Figure 7.8: Delayed reflections of radio waves.

As can be seen from the discussion in Sections 7.1.3-7.1.5, the propagation models for the wireless channel can quickly become quite complicated. As mentioned in Section 7.1.4, there can also be delays $\{\tau_d\}$ between different reflected waves (see also Figure 7.8) and hence we can get the received baseband signal to be,

$$y(t) = \sum_{d=1}^{\mathcal{D}} \underbrace{\frac{\mathcal{S}_d}{(r_d)^{\frac{\beta}{2}}} f_d^{'}(t)}_{f_d(t)} \, x(t-\tau_d) = \sum_{d=1}^{\mathcal{D}} f_d(t)x(t-\tau_d) \tag{7.12}$$

where $\mathcal{D}$ are the number of delayed paths with $\mathcal{S}_d$ and r_d respectively the shadowing and distance losses of the delayed paths. There could be several reflectors at approximately the same delay $\{\tau_d\}$, and hence as in Section 7.1.4, (7.11), we have

$$f_d^{'}(t) = \sum_{l=1}^{L_d} c_l(d) e^{j\frac{2\pi}{x} v \cos\theta_l(d)t}, \quad d = 1, ..., \mathcal{D}. \tag{7.13}$$

Adding is the usual receiver noise into (7.12), we get the received signal to be

$$y(t) = \sum_{d=1}^{\mathcal{D}} f_d(t)x(t-\tau_d) + z(t) \tag{7.14}$$

where $z(t)$ is typically additive Gaussian noise and we have defined $f_d(t) = \frac{\mathcal{S}_d}{(r_d)^{\frac{\beta}{2}}} f_d^{'}$. As can be seen, the description of the model depends crucially on the multipath fading $\{f_d(t)\}$ as well as the delays $\{\tau_d\}$ aside from the coarser quantities of distance loss and shadowing. The power of each delayed path depends on the coarser quantities which leads to the *power-delay* profile in (7.14) which is the main model for communication.

7.2.1 Linear time-varying channel

The model given is (7.14) can be represented as a linear time-varying channel as follows. We can rewrite (7.14) as,

$$y(t) = \int_{\tau} h_p(t;\tau) x(t-\tau) d\tau + z(t)$$

where

$$h_p(t;\tau) = \sum_{d=1}^{\mathcal{D}} f_d(t) \delta(\tau - \tau_d).$$

This compactly represents the linear time-varying channel imposed by the electromagnetic wave propagation. Now, we can go to the discrete time model by sampling at Nyquist rate. Note that for a time-varying system this means sampling at a rate higher than $2(W_I + W_s)$ where W_I is the input bandwidth and W_s is the bandwidth of the channel time-variation. Now if we sample at times $T_s < \frac{1}{2(W_I+W_s)}$, then

$$y(kT_s) = \int_{\tau} h_p(kT_s;\tau) x(kT_s - \tau) d\tau + z(kT_s).$$

Now, if the transmitted signal was modulated as in Chapter 4, then

$$x(t) = \sum_n x_n \varphi(t - nT)$$

Where T is the symbol period and as in Chapter 4, we have assumed $N = 1$ complex dimensions for the modulation. Therefore if we assume that $T_s = \frac{T}{L}$, where L as in Chapter 5 is the "over sampling" factor, then,

$$\begin{aligned}
y_i(k) &\triangleq y(kT - \frac{iT}{L}) = \sum_{d=1}^{\mathcal{D}} f_d(kT - \frac{iT}{L}) \sum_n x_n \varphi(kT - \frac{iT}{L} - \tau_d - nT) + \underbrace{z(kT - \frac{iT}{L})}_{z_i(k)} \\
&= \sum_n x_n \sum_{d=1}^{\mathcal{D}} f_d(kT - \frac{iT}{L}) \varphi \left[(k-n)T - \frac{iT}{L} - \tau_d \right] + z_i(k) \\
&= \sum_m x_{k-m} \left\{ \sum_{d=1}^{\mathcal{D}} f_d(kT - \frac{iT}{L}) \varphi(mT - \frac{iT}{L} - \tau_d) \right\} + z_i(k) \\
&\triangleq \sum_m x_{k-m} h_i(k;m) + z_i(k)
\end{aligned}$$

where

$$h_i(k;m) = \sum_{d=1}^{\mathcal{D}} f_d(kT - \frac{iT}{L}) \varphi(mT - \frac{iT}{L} - \tau_d)$$

Therefore a vector linear time-varying channel model is,

$$\mathbf{y}(k) = \begin{bmatrix} y_0(k) \\ \cdot \\ \cdot \\ \cdot \\ y_{L-1}(k) \end{bmatrix} = \sum_m \mathbf{h}(k;m)x_{k-m} + \mathbf{z}(k) \tag{7.15}$$

where

$$\mathbf{h}(k;m) = \begin{bmatrix} h_0(k;m) \\ \cdot \\ \cdot \\ \cdot \\ h_{L-1}(k;m) \end{bmatrix}, \quad \mathbf{z}(k) = \begin{bmatrix} z_0(k) \\ \cdot \\ \cdot \\ \cdot \\ z_{L-1}(k) \end{bmatrix} \tag{7.16}$$

The model given in (7.15) is the linear time-varying model that summarizes the effects of the radio wave propagation on the communication channel.
Notes:

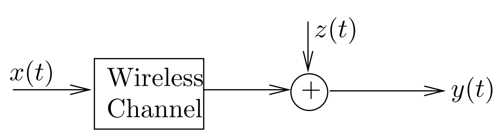
1. This model looks very similar to the one we have been studying till now in this class. The main difference is that the linear time-invariant channel has been replaced by the linear time-varying channel.

2. The impulse response $\{h(k,l)\}$ is composed of several components, due to reflection, absorption, path loss, an multipath combining. Instead of a detailed model for this, it is more fruitful to describe it statistically, as we will do in Section 7.2.2.

3. The processes $\{f_d(t)\}$ vary in time and their correlation depends on the relative velocity between sender and receiver. The bandwidths of these variations depend on the Doppler shifts of the reflected paths. Thus the random process $\{f_d(t)\}$ has a spectral bandwidth which is roughly proportional to $f_c \frac{v}{c}$. For example at $f_c = 1$ GHz, $v = 30m/s (\approx 60mph)$, the bandwidth of variation is approximately 100 Hz.

4. The time-scales become important. During a transmission burst which might last a few seconds at most, the path loss, shadowing (absorption) and the reflectors remain approximately constant. The main variation comes from Doppler spread *i.e.,* mobility. Hence, the variation in $f_d(t)$ arises mainly from this effect.

7.2.2 Statistical Models



In the model given in (7.15), one can describe in fine detail each of the quantities influencing $\mathbf{h}(k;m)$ in terms of the reflector geometries and the delays. However, as one can imagine, such a description can quickly become difficult in most realistic environments. Therefore we revert to a statistical (random) channel description that captures the spirit rather than each of the fine details.
The first question is about the behavior of $\{f_d'(t)\}$ given in (7.13), which greatly infuences the linear time-varying response $\mathbf{h}(k;m)$. Given that it corresponds to the addition of several random quantities,

one can surmise that it can be well modelled by a Gaussian stochastic process. In fact this has also been well justified by empirical studies. It has been shown that it can be well modelled as a complex Gaussian process which has a power spectral density which is relatively flat over a bandwidth of $(\frac{v}{\lambda})$ Hz, where λ is the carrier wavelength. Therefore we have the model that $f_d(t)$ is Gaussian with

$$S_{f_d'}(\omega) = \begin{cases} \frac{1}{2W_s}, & |\,\omega\,| < 2\pi\frac{v}{\lambda} \\ 0 & else \end{cases} \tag{7.17}$$

where $W_s = \frac{v}{\lambda}$. It is also a good model that the reflectors at different delays have independent fading, *i.e.,* $\{f_d'(t)\}$ are i.i.d. over $d = 1, ..., \mathcal{D}$. Therefore we have statistical description of the fading processes $\{f_d'(t)\}$.

The delays $\{\tau_d\}$ are difficult to characterize empirically. Depending on the propagation environment (urban, sub-urban or rural) the maximal delay $\tau_{\mathcal{D}}$ can be characterized. For example in urban environments, $|\,\tau_{\mathcal{D}}\,| < 5$ μsec. Where as in rural environments one could have larger delays of the order of tens of μsecs. This corresponds to reflectors being further away like mountains etc. Note that a 1 μsec delay implies a distance of 300 m ($3 \times 10^8 \times 10^{-6} = 300$). As mentioned earlier, there is a power-delay profile associated with $\{f_d'(t)\}$.Usually the larger delay reflections have lower power since they have travelled a larger distance. However the effects of shadowing means that there is randomness in the power-delay profile as well. Therefore there are the models for the delays depending on the environment and typically the models are given for urban, sub-urban or rural propagation scenarios.

Special Cases

1. *Linear time invariant model:* if $v = 0$, *i.e.,* the mobile is stationary, then $f_d(t)$ is independent of t, (see (7.14)), and we recover a linear time invariant model since $(\mathbf{h}(k;m) = \mathbf{h}(0;m)$, and hence (7.15) becomes

$$\mathbf{y}(k) = \sum_m \mathbf{h}(0;m)x_{k-m} + \mathbf{z}(k)$$

2. *Flat fading model:* If the *delay spread i.e.,* $\max_d \tau_d = \tau_{\mathcal{D}}$, is small, *i.e,* $\tau_{\mathcal{D}} \approx 0$, then we get

$$h_i(k;m) = \left[\sum_{d=1}^{\mathcal{D}} f_d(kT - \frac{iT}{L})\right] \varphi(mT - \frac{iT}{L})$$

Hence

$$h_0(k;m) = \left[\sum_{d=1}^{\mathcal{D}} f_d(kT)\right] \varphi(mT).$$

Therefore, for this case with $L = 1$, we get for pulse shapes like the sinc pulse which have

$$\varphi(mT) = 0, \quad m \neq 0$$

the flat fading model where there is no inter-symbol interference, *i.e.,*

$$y_0(k) = h_0(k;0)x_k + z_0(k).$$

This gives a frequency "non-selective" (*i.e.,* no ISI) but time varying channel.

7.2.3 Time and frequency variation

In Chapter 4-6, we encountered ISI channels where the model was a linear time invariant channel. In this chapter the model of the wireless channel is one of linear time-varying channel.
Now, the variation in time depends or the Doppler shifts as seen in (7.10), and therefore across *all* the reflected paths one can define the largest Doppler variations. Clearly this is upper bounded by $\frac{2v}{\lambda}$ and this is the maximal *Doppler spread* f_{max} of the time variation. This was the bandwidth of time variation used in (7.17) for the statistical model. This Doppler spread corresponds to how fast the channel varies over time. Hence we define the *coherence time* T_c as,

$$T_c \triangleq \frac{1}{f_{max}}$$

which corresponds approximately the time period over which the channel is expected to remain roughly the same. In a similar manner we can find the maximal delay spread of the channel response as $T_d = \max_d \tau_d$. This relates to the longest delay seen and characterizes the frequency variation of the random frequency response of the wireless channel. We define the *coherence bandwidth* W_c as

$$W_c \triangleq \frac{1}{2T_d}$$

which corresponds approximately to the frequency range over which we expect the response to be roughly the same.
Therefore we obtain characterization of the order-of-magnitude behavior in time and frequency of the random time-varying channel through the coherence time and coherence bandwidths. For example, if we are transmitting at 1 GHz, and a mobile is moving at 60 miles per hour (approximately $30\frac{m}{s}$) then we have $f_{max} = \frac{2v}{\lambda} = 200$ Hz and $T_c = 5$ msec. Therefore approximately every 5 msec the channel becomes independent and this gives us an engineering insight into block lengths we could use and other design considerations. Similarly if we have a maximum delay spread of 20 μsec, then $W_c = 2.5 \times 10^4$. Therefore for signal bandwidth of less than W_c, we would expect the variation of the channel in frequency to be small.

7.2.4 Overall communication model

The wireless model given in (7.15) forms the basis of all the discussion in Chapter 8. The model comprises of time-variation which is roughly characterized by the coherence time T_c, and frequency variation (approximately characterized by W_c, the coherence bandwidth). The frequency variation causes intersymbol interference and we have studied such impairments in Chapter 4-6. However due to time-variation one might encounter several realizations of the channel over a transmission burst and this is exactly the new feature in wireless communication. We will need to understand the impact of this on the communication system. Therefore we have a linear time-varying communication model that is modelled as a stochastic process as described in Section 7.2.2.
Block time-invariant models: In many situations it is useful to consider the channel as remaining constant over a small number of transmissions and then change to another value after that. For example if we transmit a block over a duration which is much smaller than the coherence time T_c of the channel, then for all practical purpose the channel encountered over the transmission block is constant. Then if the next transmission occurs at a time which is longer than T_c after the previous transmission, then the channel encountered can be well modelled as being independent of the previous transmission. This argument leads us to an intermediate model between a continuously varying channel and a time-invariant channel, *i.e.,* a *block time-invariant* channel.
Consider a transmission of K symbols with symbol period T. If $kT \ll T_c$, then $\{\mathbf{h}(k;n)\}$ are approximately the same for each of the symbols. If we call the transmission block b, then we can index the "constant" channel realization by $\mathbf{h}^{(b)}$.

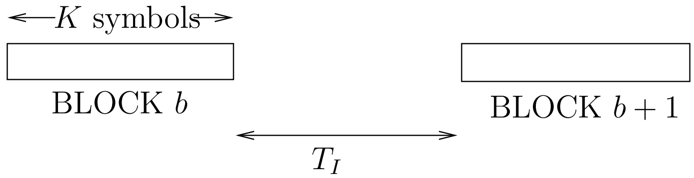


Figure 7.9: Block time-invariant model.

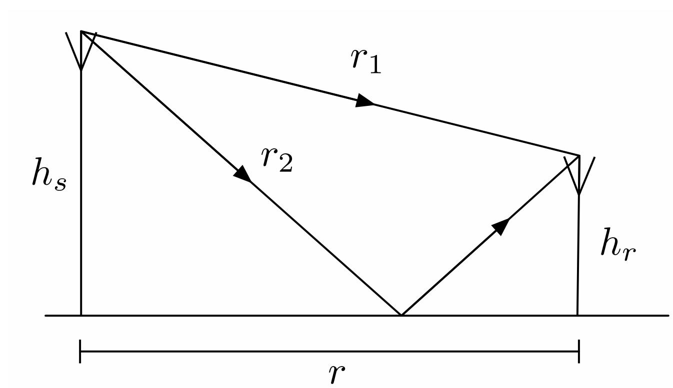
Now, if the next transmission occurs after T_I time where $T_I \gg T_c$, then the channel realization $\mathbf{h}^{(b+1)}$ is well modelled as being independent of $\mathbf{h}^{(b)}$ and identically distributed. Thus behavior is called a block time-invariant channel model.
Multiuser channels: In a multiuser setting of Chapter 9, we could have several users which have independent channels transmitting simultaneously to a single receiver. This model would a multiuser generalization to the point-to-point model given in (7.15). Such a generalization is the second distinguishing element of wireless channels and we will study the impact of multiuser interference in more detail in Chapter 9.

7.3 Problems

Problem 7.1

Suppose $g(t, \tau)$ represents the response of a linear filter at time t to an impulse τ time units earlier, i.e., if the signal x is the input to the filter its output y is given by $y(t) = \int x(t-\tau)g(t,\tau)d\tau$. How can we tell if the filter described by g is time-invariant?
A linear time invariant filter can be characterized by its impulse response $h(t)$, which is the response at time t to an impulse at time 0. We can view this linear time invariant filter as just a linear filter with response $g(t, \tau)$. What is the relationship between $g(\tau, t)$ and $h(t)$ (i.e., what is g in terms of h)?

Problem 7.2



1. Let r_1 be the length of the direct path and r_2 the length of the reflected path (from the transmitter to the ground and from the ground to the receiver). Show that $r_2 - r_1$ is asymptotically equal to a/r. Find a.

2. Assume that the received waveform at the receiving antenna is given by

$$E_r(f,t) = \frac{\Re[\alpha \exp\{j2\pi[ft - fr_1/c]\}]}{r_1} - \frac{\Re[\alpha \exp\{j2\pi[ft - fr_2/c]\}]}{r_2}$$

   Approximate the denominator r_2 by r_1 and show that $E_r \approx \frac{\gamma}{r^2}$ for $1/r \ll c/f$. Find γ.

3. Explain why the asymptotic expression remains valid without first approximating the denominator r_2 by r_1.

Problem 7.3

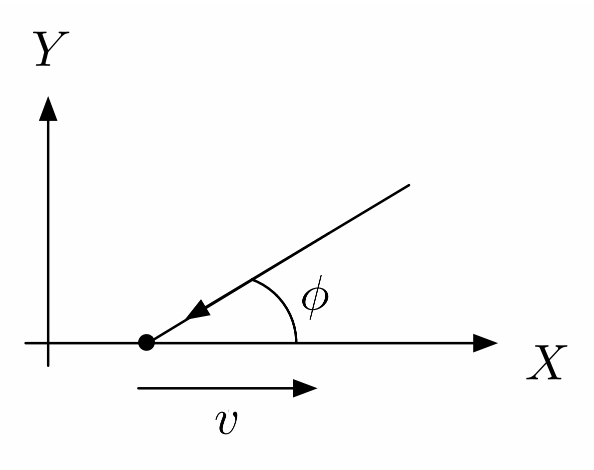


Figure 7.10: Moving mobile receiver

Consider a mobile receiver moving along the X-axis with velocity v as shown in Figure 7.10. The received signal arrives at an angle ϕ. What is the amount of associated Doppler shift?

Chapter 8

Single-user communication

In wireless channels the detection problem is different from that considered for linear time-invariant channels (studied in Chapter 4-5) in two important ways.
First as seen in Chapter 7, the wireless channel can vary quite rapidly (within a few milliseconds) and therefore one might not be able to have an accurate estimate of the channel. This is because the channel estimation schemes (such as those studied in Section 6.3) might not have enough training symbols to accurately estimate the rapidly varying channel. Therefore, we might have to design detectors that do not utilize the actual realization of the channel but only the know statistics of the channel. Such detectors are called *non-coherent* detectors as opposed to *coherent* detectors which use channel knowledge (or estimates) for detection. We will study non-coherent detectors in Section 8.1.2.
The second difference is in the error probability behavior. In the linear time invariant channels studied in Chapter 4-5, the error probability was averaged over the noise statistics alone. This was because the linear time-invariant channel was the same over the entire duration of the transmission. However, in wireless communication the channel need not be a constant over the duration of the transmission. Consider the block time-invariant model introduced in Section 7.2.4.
Given any transmission strategy and a detection strategy, clearly the error probability averaged over the noise statistics depends on the channel realization $\{\mathbf{h}(k;n)\}$, *i.e.,*

$$P_{e,b}(\hat{\mathbf{x}}_b \neq \mathbf{x}_b) = g(\mathbf{h}^{(b)}; SNR)$$

where $g(\cdot,\cdot)$ is a function of $\mathbf{h}^{(b)}$, the channel realization for transmission block b, $\{(\mathbf{h}^{(b)})\}$ as well as the SNR. Now if we transmit over B such blocks, the average error block error probability is

$$P_e(\{\hat{\mathbf{x}}\} \neq \{\mathbf{x}\}) = \frac{1}{B}\sum_{b=1}^{B} P_{e,b}(\hat{\mathbf{x}}_b \neq x_b).$$

If the channel realization process $\{\mathbf{h}^{(b)}\}$ is assumed to be i.i.d. over transmission blocks (or more generally stationary and ergodic), then over a large number of blocks (*i.e.,* $B \to \infty$), the average block error probability would go to

$$\bar{P}_e = \mathbb{E}_{\mathbf{h}_b}[P_{e,b}(\hat{x}_b \neq x_b)].$$

Therefore there is a second averaging we need to perform for wireless channels which are over the channel statistics and this is a new feature we will encounter. As we will see in Section 8.1, this will considerably alter the behavior of the error probability with SNR.
In Section 8.1 we will study coherent and non-coherent detectors for the channel given in (8.1) and also find the behavior of the error probability with SNR. In Section 8.2 we will study the tool of diversity which is an important method to deal with the fading wireless channel.

8.1 Detection for wireless channels

In this section we will derive and analyse the average block error probability of detection for wireless channels. In Section 8.1.1, we analyse the coherent detector, where the channel varies slowly enough for us to "perfectly" estimate the channel and use the realization for detection. This is similar in flavor to detection schemes studied earlier in Chapters 2 and 4. However, when we have fast time-varying channels we may not have knowledge of the channel realization but only its statistical behavior. In such cases we have a non-coherent detection scheme as described in Section 8.1.2.
In the following we will for simplicity assume a flat fading model (see Section 7.2.2) where the baseband received signal in transmission block b, is well modeled as,

$$y(k) = h^{(b)}x(k) + z(k),, \quad k \in \mathcal{T}(b) \tag{8.1}$$

with $h^{(b)}$ being a complex Gaussian process which is i.i.d. over transmission blocks b. For notational reasons we will denote the time span of the b^{th} block by $\mathcal{T}(b)$.

8.1.1 Coherent Detection

Suppose we estimate the channel and use this to detect the transmitted symbols. This is suitable if the channel was block time-invariant (as given in (8.1), also see Section 7.2.4). Therefore one could insert training symbols to estimate the particular channel realization for the block. If the channel varies slowly, we can get a very accurate estimate of it. Therefore we assume that the channel is known *perfectly* at the decoder. That is, for the maximum-likelihood decoding, for block b

$$y(k) = h^{(b)}x(k) + z(k), \quad k \in \mathcal{T}(b)$$

we get

$$\hat{\mathbf{x}}_b = \arg\max_{\mathbf{x}} \mathbb{P}_{\mathbf{Y}|\mathbf{X},h}(\mathbf{y}|\mathbf{x}, h^{(b)})$$

where $\{\hat{\mathbf{x}}_b\} = \{\hat{x}(k)\}$ for the b^{th} block. Therefore the decision rule is

$$\hat{\mathbf{x}}_b = \arg\min_{\mathbf{x}} \sum_{k\in\mathcal{T}(b)} \left| y(k) - h^{(b)}x(k) \right|^2$$

since $\{z(k)\}$ is AWGN.
As mentioned earlier, the probability of error is calculated by averaging the error probability over many transmission blocks. For a given transmission block b, the error probability for binary transmission is

$$P_e(h^{(b)}) = Q\left(\frac{d_{min}|h^{(b)}|}{2\sigma_z}\right), \tag{8.2}$$

as was derived in Chapter 2 where we notice that the error depends on minimum distance of the altered constellation $h^{(b)}x(k)$.
It is well known that the $Q(\cdot)$ function decays exponentially, *i.e.,*

$$\frac{1}{2\sqrt{2\pi x}}\left[1 - \frac{1}{x^2}\right] e^{-\frac{x^2}{2}} \le Q(x) \le e^{-\frac{x^2}{2}}, \quad x > 1$$

Therefore we have

$$P_e(h^{(b)}) = Q\left(\frac{d_{min}|h^{(b)}|}{2\sigma_z}\right) \le e^{-\frac{d_{min}^2|h^{(b)}|^2}{8\sigma_z^2}}. \tag{8.3}$$

Note that $P_e(h^{(b)})$ is a random variable depending on the random channel realization $h^{(b)}$. In order to find the error probability we need to calculate

$$\bar{P}_e = \mathbb{E}_h Q\left(\frac{d_{min}|h^{(b)}|}{2\sigma_z}\right) \le \mathbb{E}_h e^{-\frac{d_{min}^2 |h^{(b)}|^2}{8\sigma_z^2}} \tag{8.4}$$

Lemma 8.1.1. *If* $U \sim \mathbb{C}\mathcal{N}(0, \mathbf{K}_U)$, $(\mathbf{K}_U^{-1} + b\mathbf{A}) > 0$, *and* $\mathbf{A} = \mathbf{A}^*$, *for some* $b \in \mathbb{C}$ *then*

$$\mathbb{E}_U \left[e^{-b\mathbf{U}^*\mathbf{A}\mathbf{U}}\right] = \frac{1}{|\mathbf{I} + b\mathbf{K}_U\mathbf{A}|} \tag{8.5}$$

Proof:

$$\begin{aligned}
\mathbb{E}_U \left[e^{-b\mathbf{U}^*\mathbf{A}\mathbf{U}}\right] &= \frac{1}{\pi}\int_{\mathbf{U}} \frac{1}{\mid \mathbf{K}_U \mid} e^{-\mathbf{U}^*\mathbf{K}_U^{-1}\mathbf{U}} e^{-b\mathbf{U}^*\mathbf{A}\mathbf{U}} d\mathbf{U} \\
&= \frac{1}{\pi}\frac{1}{\mid \mathbf{K}_U \mid}\int_{\mathbf{U}} e^{-\mathbf{U}^*(\mathbf{K}_U^{-1}+\mathbf{A}b)\mathbf{U}} d\mathbf{U}
\end{aligned}$$

Let $\tilde{\mathbf{K}}_U = (\mathbf{K}_U^{-1} + b\mathbf{A})^{-1}$, then

$$\begin{aligned}
\mathbb{E}_U \left[e^{-b\mathbf{U}^*\mathbf{A}\mathbf{U}}\right] &= \frac{1}{\pi}\frac{1}{|\mathbf{K}_U|}\int_{\mathbf{U}} e^{-\mathbf{U}^*(\mathbf{K}_U^{-1}+\mathbf{A}b)\mathbf{U}} d\mathbf{U} \\
&= \frac{|\tilde{\mathbf{K}}_U|}{|\mathbf{K}_U|}\left\{\frac{1}{\pi}\frac{1}{|\tilde{\mathbf{K}}_U|}\int_{\mathbf{U}} e^{-\mathbf{u}^*\tilde{\mathbf{K}}_U^{-1}\mathbf{U}} d\mathbf{U}\right\} \\
&= \frac{|\tilde{\mathbf{K}}_U|}{|\mathbf{K}_U|} = \frac{1}{\left|\mathbf{K}_U(\mathbf{K}_U^{-1} + b\mathbf{A})\right|} \\
&= \frac{1}{|\mathbf{I} + b\mathbf{K}_U\mathbf{A}|}
\end{aligned}$$

□

Now, we use this result in (8.4) to get an upper bound to the average error probability. By identifying $u = h^{(b)}$, $A = \frac{d_{min}^2}{8\sigma_z^2}$, $b = 1$ and $\mathbf{K}_U = 1 = \mathbb{E}[|\mathbf{h}^{(b)}|^2]$, we get

$$\bar{P}_e \le \mathbb{E}_h\left\{e^{-\frac{d_{min}^2|h|^2}{8\sigma_z^2}}\right\} = \frac{1}{1 + \frac{d_{min}^2}{8\sigma_z^2}} \tag{8.6}$$

Therefore in BPSK transmission, with average energy $\mathcal{E}_x$, we have $d_{min} = 2\sqrt{\mathcal{E}_x}$ and hence we have,

$$\bar{P}_e \le \frac{1}{1 + SNR} \le \frac{1}{SNR} \tag{8.7}$$

and hence the behavior of the error probability at high SNR is like $\frac{1}{SNR}$. This can be seen more explicitly by carrying out the calculation in (8.4) exactly.
This calculation is done in Appendix 8.A and the result from (8.30) is that,

$$\bar{P}_e = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1 + \frac{8\sigma^2}{d_{min}^2}}}\right].$$

Now, if we have BPSK, where $\pm\sqrt{\mathcal{E}_x}$ are the symbols, this gives $d_{min} = 2\sqrt{\mathcal{E}_x}$ and hence $\frac{d_{min}^2}{\sigma_z^2 8} = \frac{\mathcal{E}_x}{2\frac{N_0}{2}} = \frac{\mathcal{E}_x}{N_0} = SNR$.

Hence we have

$$\bar{P}_e = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1+\frac{1}{SNR}}}\right] \tag{8.8}$$

Now, at high SNR, *i.e.,* $SNR \to \infty$, we have

$$\bar{P}_e = \frac{1}{2}\left[1-(1+SNR^{-1})^{-\frac{1}{2}}\right] \approx \frac{1}{2}\left[1-(1-\frac{1}{2}\frac{1}{SNR})\right]$$

This gives at high SNR

$$\bar{P}_e \approx \frac{1}{4SNR},$$

which is the same behavior as was observed in (8.6), where we just did the upper bound.
From this calculation we learn two important lessons. One is that at high SNR, the average error probability decays inversely with SNR instead of exponentially as in an AWGN channel. The second lesson is that the upper bound technique developed in Lemma 8.1.1 is actually tight at high SNR.
The first observation is quite striking since it means that to obtain the same error rates, one needs to go to much higher SNR in a fading channel as compared to that in a time invariant channel. The surprise is that this behavior occurs even with a coherent receiver which uses *perfect* channel information. A rough intuitive argument for this can be seen as follows. From (8.2) we see that for a single frame the error probability is given by

$$P_e(h^{(b)}) = Q\left(\frac{d_{min}}{2\sigma_z}\,|h^{(b)}|\right) = Q\left(\sqrt{2SNR}\,|h^{(b)}|\right)$$

and hence if $|h^{(b)}|\sqrt{2SNR} \gg 1$, then $P_e(h^{(b)}) \approx 0$ and if $|h^{(b)}|\sqrt{2SNR} \ll 1$, then $P_e(h^{(b)}) \approx \frac{1}{2}$. Therefore a frame is in error with high probability when the channel gain $|h^{(b)}|^2 \ll \frac{1}{SNR}$, *i.e.,* when the channel is in a "deep fade". Therefore the average error probability is well approximated by the probability that $|h^{(b)}|^2 \ll \frac{1}{SNR}$, and for high SNR we can show that

$$\mathbb{P}\left\{|h|^2 < \frac{1}{SNR}\right\} \approx \frac{1}{SNR}$$

and this explains the behavior of the average error probability. Although this is a crude analysis, it brings out the most important difference between the AWGN and the fading channel. The typical way in which an error occurs in a fading channel is due to channel failure, *i.e.,* when the channel gain $|h|$ is very small, less than $\frac{1}{SNR}$. On the other hand in an AWGN channel errors occur when the noise is large and since the noise is Gaussian, it has an exponential tail causing this to be very unlikely at high SNR.

8.1.2 Non-coherent Detection

As mentioned before, due to the fast time variation, one could have very imperfect knowledge of the channel state $h^{(b)}$ in a particular block b. This could occur if the block is too short and/or if the channel varies too fast. In such a case we can take an extremal view where we have *no knowledge* of the channel at the receiver, but only know its statistical behavior (*i.e.,* that $h^{(b)} \sim \mathbb{C}\mathcal{N}(0,1)$ and that it is i.i.d. from block to block). In such a case the detector structure becomes different from the coherent detector studied in Section 8.1.1.
Let us consider a block time invariant model where

$$y(k) = h^{(b)}x(k) + z(k), \quad k \in \mathcal{T}(b)$$

Let $h^{(b)} \sim \mathbb{C}\mathcal{N}(0,1)$ and let us make K observations, *i.e.*, a transmission block of K symbols ($|\mathcal{T}(b)| = K$). If we collect these K samples in a vector form it gives us

$$\mathbf{y} = \begin{bmatrix} y(K-1) \\ \cdot \\ \cdot \\ \cdot \\ y(0) \end{bmatrix} = h \underbrace{\begin{bmatrix} x(K-1) \\ \cdot \\ \cdot \\ \cdot \\ x(0) \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z(K-1) \\ \cdot \\ \cdot \\ \cdot \\ z(0) \end{bmatrix}}_{\mathbf{z}}$$

Hence we get for the likelihood function

$$P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}) \sim \mathbb{C}\mathcal{N}(0, \mathbb{E}[|h|^2]\mathbf{x}\mathbf{x}^* + \sigma_z^2 \mathbf{I}_K).$$

Hence for $\mathbb{E}[|h|^2] = 1$, the log-likelihood function is

$$\log P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y} \mid \mathbf{x}) = -\log \pi - \log |\mathbf{x}\mathbf{x}^* + \sigma_z^2 \mathbf{I}_K| - \mathbf{y}^*(\mathbf{x}\mathbf{x}^* + \sigma_z^2 \mathbf{I}_K)^{-1}\mathbf{y} \tag{8.9}$$

Now,

$$|\mathbf{x}\mathbf{x}^* + \sigma_z^2 \mathbf{I}_K| = (\sigma_z^2)^K \mid \frac{\mathbf{x}\mathbf{x}^*}{\sigma_z^2} + \mathbf{I}_K \mid \overset{(a)}{=} \left(\frac{\mathbf{x}^*\mathbf{x}}{\sigma_z^2} + 1 \right) (\sigma_z^2)^K$$

where (a) follows because $\mid \mathbf{I} + \mathbf{AB} \mid = \mid \mathbf{I} + \mathbf{BA} \mid$.
Assumption: Now if we use a constant modulus transmission, *i.e.*, $|x(i)|^2 = \mathcal{E}_x$, which occurs for example with PSK constellations, we get

$$|\mathbf{x}\mathbf{x}^* + \sigma_z^2 \mathbf{I}_K| = (\sigma_z^2)^K \left[1 + \frac{K\mathcal{E}_x}{\sigma_z^2} \right]$$

which is independent of $\mathbf{x}$. Therefore for the constant modulus constellation the maximum likelihood criterion from (8.9) gives us

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \left\{ -\mathbf{y}^*(\mathbf{x}\mathbf{x}^* + \sigma_z^2 \mathbf{I}_K)^{-1}\mathbf{y} \right\} \tag{8.10}$$

Now, if we use the matrix inversion lemma, Lemma 5.7.1, we have

$$\begin{aligned} \left[\mathbf{x}\mathbf{x}^* + \sigma_z^2 \mathbf{I}_K\right]^{-1} &= \frac{1}{\sigma_z^2}\mathbf{I}_K - \frac{1}{\sigma_z^2}\mathbf{x}\left[1 + \frac{\mathbf{x}^*\mathbf{x}}{\sigma_z^2}\right]^{-1} \frac{1}{\sigma_z^2}\mathbf{x}^* \\ &= \frac{1}{\sigma_z^2}\left\{ \mathbf{I}_K - \frac{1}{\sigma_z^2} \frac{\mathbf{x}\mathbf{x}^*}{1 + \frac{\mathbf{x}^*\mathbf{x}}{\sigma_z^2}} \right\} \\ &\overset{(b)}{=} \frac{1}{\sigma_z^2}\left\{ \mathbf{I}_K - \frac{1}{\sigma_z^2} \frac{\mathbf{x}\mathbf{x}^*}{1 + \frac{K\mathcal{E}_x}{\sigma_z^2}} \right\} \end{aligned} \tag{8.11}$$

where (b) follows due to the constant modulus assumption *i.e.*, $|x(i)|^2 = \mathcal{E}_x, \forall i$. Using (8.11) in (8.10) the detection criterion becomes,

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \mathbf{y}^*\mathbf{x}\mathbf{x}^*\mathbf{y} = \arg\max_{\mathbf{x}} |\mathbf{y}^*\mathbf{x}|^2 \tag{8.12}$$

Therefore we choose the input $\mathbf{x}$ which has the largest projection (inner product) to the observation $\mathbf{y}$. This is a useful interpretation that generalizes to several other non-coherent detection schemes. An illustration of a fast time-varying case for the non-coherent detector is given in Appendix 8.B.

Now, in Appendix 8.C we derive the error probability behavior of the detector given in (8.12).
As seen in Appendix 8.C, even for the non-coherent case we get an error probability that decays as $\frac{1}{SNR}$ at high SNR. Hence at high SNR, the main cause of error is a "bad" channel realization h. This is the main cause of the striking difference between fading channels and AWGN channels. A natural question to ask is weather there is a mechanism by which we can improve the performance of transmission over fading channels? This is what we study in Sections 8.2-8.4.

8.1.3 Error probability behavior

As has been seen in Section 8.1.1-8.1.2, and the detailed calculations in Appendices 8.A and 8.C, we see that the error probability for both the coherent and the non-coherent detectors behave inversely with SNR, at high SNR.
As mentioned earlier, this is in contrast to the AWGN channel where the error probability decreases exponentially with SNR. To get a feel for the difference, consider an AWGN channel with a $\frac{d_{min}}{2\sigma} = 13.5$ dB, then the error probability is

$$P_e = Q(10^{1 \cdot \frac{35}{2}}) \approx 10^{-6}$$

where as for a coherent detector we see from (8.8) that

$$\bar{P}_e = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1 + 10^{-1 \cdot \frac{35}{\sqrt{2}}}}}\right] \approx 0.01$$

Therefore the error probability is orders of magnitude larger. In fact to get to an error rate of 10^{-6}, for the fading channel one would need an SNR of over 60 dB! This shows that for reliable communication one fading channels one would require other tools. These tools are called diversity which we study in Section 8.1.4.
Diversity order: In both the coherent and non-coherent cases we were interested in the error probability behavior in the high SNR regime. In order to compare the behavior of different systems in this regime, we use the notion of diversity order which is defined below.

Definition 8.1.1. *A coding scheme which has an average error probability $\bar{P}_e(SNR)$ as a function of SNR that behaves as*

$$\lim_{SNR \to \infty} \frac{\log \bar{P}_e(SNR)}{\log(SNR)} = -d$$

is said to have a diversity order of d.

In words, a scheme with diversity order d has an average error probability of the order $\frac{1}{SNR^d}$ at high SNR. We would also use the notation

$$\bar{P}_e(SNR) \doteq SNR^{-d}$$

to indicate such a relationship where we do not explicitly specify the constants involved.

8.1.4 Diversity

Fading channels imply that there could be a high probability of "link" failure. Therefore a natural approach is to send over several links each of which fail independent of one another. This is the basic idea of diversity, where we send information over links such that the probability of failure of all links is exponentially smaller than the fading of a single one. The main question is how such a mechanism changes performance and can we build transmission and reception schemes that utilize the presence of multiple conduits of information transmission.

There are several ways in which we can obtain diversity. One is over time, so that the channel fade realization varies independently. For example by transmitting symbols separated by the coherence time of the channel, the symbols experience independent fades.
A second mechanism is through frequency. In a frequency-selective channel (*i.e.* ISI channel) the path gains (fades) of the delayed paths could be (roughly) independent. Therefore this translates for the property that the frequency response of the channel is independent, if separated further than the "coherence bandwidth" of the channel.
A third mechanism is through space, *i.e.,* when the signal is transmitted (and/or received) through multiple antennas. Since we saw that the response of the electromagnetic propagation depends on the spatial location, the spatial diversity samples this response at multiple locations by using multiple antennas. If the antennas are separated by distance larger than the "spatial coherence distance" then one would expect (almost) independent fading at the different antennas.

8.2 Time Diversity

Here we transmit information spread over a time period *larger* than the coherence time of the channel. Typically, we "interleave" the transmitted symbols so that they experience independent fading. An interleaver just permutes the transmitted symbols.

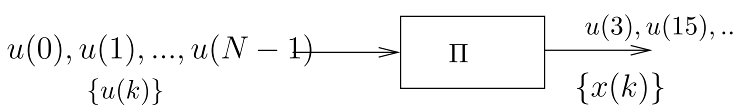


Figure 8.1: Interleaving operation.

For example for a given block size N, the interleaver chooses a particular one of $N!$ choices of permutation, and therefore the operation can be represented as $x(k) = u(\pi(k))$. The goal is to "uniformly" intersperse the symbols in the block. If the interleaver is effective, the consecutive information symbols would be dispersed far enough to experience independent fading. If this occurs, then we get

$$y(k) = h(k)x(k) + z(k),\ k = 0, \dots, \mathcal{D} - 1 \tag{8.13}$$

and $\{h(k)\}$ are i.i.d., *i.e.,* we have independent fading over $\mathcal{D}$ symbols. Hence we are transmitting over $\mathcal{D}$ independent instantiations of the fading and have created *time diversity.* The question we address next is how to utilize the time diversity we have created.

8.2.1 Repetition Coding

The simplest way one can utilize the diversity if by repeating the same symbol over all the diversity paths. For example, in (8.13), we could send the same symbol $x(0)$ for all times $k = 1, \dots, \mathcal{D} - 1$. Then

$$y(k) \quad = \quad h(k)x(0) + z(k),\ k = 0, \dots, \mathcal{D} - 1$$

This yields,

$$\underbrace{\begin{bmatrix} y(\mathcal{D}-1) \\ \vdots \\ y(0) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} h(\mathcal{D}-1) \\ \vdots \\ h(0) \end{bmatrix}}_{\mathbf{h}} x(0) + \underbrace{\begin{bmatrix} z(\mathcal{D}-1) \\ \vdots \\ z(0) \end{bmatrix}}_{\mathbf{z}}. \tag{8.14}$$

Let us examine the probability that $x'(0) \neq x(0)$ is chosen by a coherent detector for (8.14), for a *given* channel realization $\mathbf{h}$. If we denote this by $\mathbb{P}(x(0) \to x'(0)|\mathbf{h})$, we then have

$$\begin{aligned}\mathbb{P}(x(0) \to x'(0)|\mathbf{h}) &= \mathbb{P}\left[||\mathbf{y} - \mathbf{h}x'(0)||^2 < ||\mathbf{y} - \mathbf{h}x(0)||^2\mathbf{h}\right] \\ &= Q\left(\frac{||\mathbf{h}||d_{min}}{2\sigma}\right) \end{aligned} \tag{8.15}$$

where d_{min} is the minimum distance of the constellation. Now, $||\mathbf{h}||^2 = \sum_{l=0}^{L-1} |h(l)|^2$, and if $\{h(l)\}$ are i.i.d. complex Gaussian, *i.e.,*

$$h(l) \sim \mathbb{C}\mathcal{N}(0,1)$$

then $||\mathbf{h}||^2$ distribution can be found to be,

$$f_{||\mathbf{h}||^2}(\alpha) = \frac{1}{(\mathcal{D}-1)!}\alpha^{\mathcal{D}-1}e^{-\alpha}, \quad \alpha \geq 0$$

Therefore to find the error probability for repetition coding, we would average (8.15) over the distribution of $||\mathbf{h}||^2$:

$$\begin{aligned}\mathbb{P}(x(0) \to x'(0)) &= \mathbb{E}_{[||\mathbf{h}||^2]}\left[\mathbb{P}(x(0) \to x'(0)|\mathbf{h})\right] \\ &= \mathbb{E}_{[||\mathbf{h}||^2]} Q\left(||\mathbf{h}||\frac{d_{min}}{2\sigma}\right) \\ &\leq \mathbb{E}_{[||\mathbf{h}||^2]} e^{-||\mathbf{h}||^2 d_{min}^2/8\sigma^2} \\ &= \int_\alpha \frac{1}{(\mathcal{D}-1)!}\alpha^{\mathcal{D}-1}e^{-\alpha}e^{-\alpha\frac{d_{min}^2}{8\sigma^2}}d\alpha \\ &= \frac{1}{(\mathcal{D}-1)!}\int_\alpha \alpha^{\mathcal{D}-1}e^{-\alpha\left(1+\frac{d_{min}^2}{8\sigma^2}\right)}d\alpha \\ &= \frac{1}{\left(1+\frac{d_{min}^2}{8\sigma^2}\right)^{\mathcal{D}}}\end{aligned}$$

Note that we could have explicitly evaluated the error probability as we did in Appendix 8.A. However, since we are interested in high SNR behavior we use the Chernoff upper bound as done in (8.7). At high SNR this upper bound accurately depicts the error probability behavior.
If $\frac{d_{min}^2}{8\sigma^2} = SNR\left(\frac{d'^2_{min}}{4}\right)$ then,

$$\begin{aligned}\mathbb{P}(x(0) \to x'(0)) &\leq \frac{1}{\left(1+SNR\frac{d'^2_{min}}{4}\right)^{\mathcal{D}}} \\ &\leq \frac{4^{\mathcal{D}}}{SNR^{\mathcal{D}}(d'^2_{min})^{\mathcal{D}}} \doteq SNR^{-\mathcal{D}}\end{aligned}$$

Hence, by using $\mathcal{D}$ diversity paths and sample repetition coding one changes the error probability behavior (at high SNR) from $\frac{1}{SNR}$ to $\frac{1}{SNR^{\mathcal{D}}}$. This is really the value of using diversity paths and this is one of the best tools available to combat fading in wireless communications.

8.2.2 Time diversity codes

Suppose we do not restrict ourselves to repetition coding, then we would return to (8.13) to rewrite it as

$$y(k) = h(k)x(k) + z(k),\ \ k = 0, \dots, \mathcal{D} - 1$$

Now, we wish to find the criteria for choosing $\{x(k)\}$ to maximize performance (lower error probability).

$$\underbrace{\begin{bmatrix} y(\mathcal{D}-1) \\ \vdots \\ y(0) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} x(\mathcal{D}-1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & x(0) \end{bmatrix}}_{\mathbf{X}} \underbrace{\begin{bmatrix} h(\mathcal{D}-1) \\ \vdots \\ h(0) \end{bmatrix}}_{\mathbf{h}} + \underbrace{\begin{bmatrix} z(\mathcal{D}-1) \\ \vdots \\ z(0) \end{bmatrix}}_{\mathbf{z}} \tag{8.16}$$

which can be written as

$$\mathbf{y} \;\; = \;\; \mathbf{X}\mathbf{h} + \mathbf{z} \tag{8.17}$$

Therefore for a block of transmissions, we can write the probability of a coherent detector erroneously $\mathbf{x}^{'}$ instead of $\mathbf{x}$ for a given realization of the channel $\mathbf{h}$ (see also (8.15) for a similar argument) is given by,

$$\mathbb{P}(\mathbf{X} \to \mathbf{X}'|\mathbf{h}) = Q\left[\left(\frac{\mathbf{h}^*(\mathbf{X}-\mathbf{X}')^*(\mathbf{X}-\mathbf{X}')\mathbf{h}}{4\sigma^2}\right)^{1/2}\right]$$

If $\mathbf{h} \sim \mathbb{C}\mathcal{N}(0,1)$ then again we can write the average pairwise error probability as,

$$\begin{aligned}
\mathbb{P}(\mathbf{X} \to \mathbf{X}') &= \mathbb{E}_{\mathbf{h}}\left[\mathbb{P}(\mathbf{X} \to \mathbf{X}'|\mathbf{h})\right] \\
&= \mathbb{E}_{\mathbf{h}} Q\left[\left(\frac{\mathbf{h}^*(\mathbf{X}-\mathbf{X}')^*(\mathbf{X}-\mathbf{X}')\mathbf{h}}{4\sigma^2}\right)^{1/2}\right] \\
&\overset{(a)}{\leq} \mathbb{E}_{\mathbf{h}} e^{-\left[\mathbf{h}^*(\mathbf{X}-\mathbf{X}')^*(\mathbf{X}-\mathbf{X}')\mathbf{h}\right]/8\sigma^2} \\
&\overset{(b)}{=} \frac{1}{\left|\mathbf{I} + \frac{(\mathbf{X}-\mathbf{X}')^*(\mathbf{X}-\mathbf{X}')}{4\sigma^2}\right|} \\
&= \frac{1}{\prod_{l=0}^{\mathcal{D}-1}\left(1 + \frac{|x(l)-x'(l)|^2}{4\sigma^2}\right)} \\
&\leq \frac{1}{\prod_{l=0}^{\mathcal{D}-1} \frac{|x(l)-x'(l)|^2}{4\sigma^2}}
\end{aligned}$$

where (a) follows because of (8.3) and (b) follows by use of Lemma 8.1.1.
Now if, $\{l : x(l) \neq x'(l)\} = \mathcal{B}$, then

$$\mathbb{P}(\mathbf{X} \to \mathbf{X}') \leq \frac{1}{\prod_{l\in\mathcal{B}}\left(1 + \frac{|x(l)-x'(l)|^2}{4\sigma^2}\right)} \doteq \frac{C}{(SNR)^{|\mathcal{B}|}}$$

Therefore the error probability behavior is dictated by $\mathcal{B}$, which is the set over which the constellation symbols differ over the diversity paths. In order to get maximal decay rate of error probability, we want $|\mathcal{B}| = \mathcal{D}$, *i.e.,* the transmitted symbols must be coded such that the distinct codewords differ on *all* diversity paths. Clearly the repetition code is a simple instance of this where distinct transmitted symbols would obviously differ on all diversity paths. However, the repetition code does so by sacrificing rate of transmission. Therefore the question is whether one can design codes which have higher transmission rate but obtain same error performance (at least the decay rate of error probability with SNR) as the repetition code. We will see that later by using spatial diversity.

8.3 Frequency Diversity

This is useful in a scenario where the transmission bandwidth is larger than the "coherence bandwidth" of the channel. This form of diversity is the dual of time diversity in that it utilizes the frequency variation of the channel (instead of time variations). The frequency variation arises because the multiple paths of the electromagnetic scattering (propagation) environment could be received at different delays. Therefore, if the fading on the different delayed paths are independent, one can obtain diversity by combining the (independent) delayed paths. Therefore in principle, they are identical to the time and spatial diversity methods. The reason why the delayed paths could experience independent fading is because, they encounter different scatterers (reflections) and hence the statistical variations (fading) would most likely be independent for different scatterers.
Hence, if there exists a "delay spread" it leads to an intersymbol interference (ISI) channel which we have studied in great depth in the earlier part of this class.
There are primarily three ways of exploiting frequency diversity:

1. **Time-domain equalization:** Here optimal receivers that we have studied for ISI channels as well as sub-optimal equalizers can be shown to take advantage of frequency diversity.

2. **Multicarrier techniques (DMT/OFDM):** In wireless channels, if the transmission block is small enough, then the channel is (approximately) constant over one transmission block. Therefore, in this block time-invariant model (studied in Section 7.2.4), the DMT/OFDM technique is able to create parallel channels in the frequency domain. If the (fading) coherence bandwidth is smaller than the overall transmission bandwidth, *i.e.*, the channel (frequency) coefficients are independent if separated by a "large-enough" frequencies, (*i.e.,* separated by the coherence bandwidth) we get frequency diversity.

3. **Spread spectrum technique:** Here the information is spread across a larger transmission bandwidth by using an operation involving pseudo-random noise sequences. Here parts of the channel spectrum which have high gain could be used to decode transmitted information. We will go into this in more detail in Chapter 9 in the context of multiple access channels.

Illustration of frequency diversity

The channel frequency selectivity is determined by the multipath delay spread, T_d introduced in Section 7.2.3. It is the bandwidth beyond which the channel response (in the frequency domain) are approximately independent. A measure of this is the coherence bandwidth, *i.e.*,

$$W_c = \frac{1}{2T_d}$$

It can also be related to the correlation between channel responses at two different frequencies, *i.e.* if

$$\Psi(\Delta f) = \mathbb{E}[H(f)H^*(f+\Delta f)]$$

the bandwidth of $\Psi(\cdot)$ is the coherence bandwidth of the channel.
Now, consider an OFDM scenario for the block time invariant channel, with N carriers,

$$Y(l) = H\left(\frac{2\pi}{N}l\right)X(l) + Z(l), \quad l = 0, \dots, N-1$$

Now, if we have a coherence bandwidth of W_c, then the frequency responses $\left\{H\left(\frac{2\pi}{N}l\right)\right\}$ which are separated by larger than this bandwidth are approximately independent. If total transmission bandwidth is

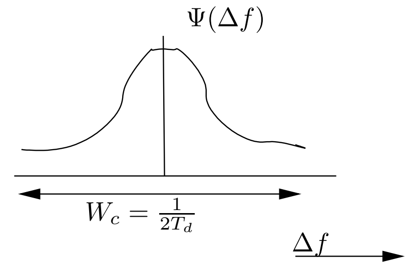


Figure 8.2: Coherence bandwidth of ISI channel.

W, and we have N OFDM tones, then the frequency is divided into $\frac{W}{N}$ sized frequency bins. Hence, a separation by K_c tones would ensure independence, where

$$K_c = \frac{W_c}{W/N} = \frac{NW_c}{W}$$

Therefore we expect,

$$\mathbb{E} H\left(\frac{2\pi}{N} l\right) H^*\left(\frac{2\pi}{N}(l + K_c)\right) \approx 0$$

Thus if we divide the frequency into $\frac{W}{W_c}$ frequency bins of size W_c each, then we have $\left(\frac{W}{W_c}\right)$ diversity paths available to us. If we denote $\mathcal{D} = \frac{W}{W_c}$, then we have created $\mathcal{D}$ independent instantiations of the random frequency response. Therefore we can use a model like (8.14) (or (8.15) for higher rates) to get an error probability behavior of $\frac{1}{SNR^{\mathcal{D}}}$ (or $\frac{1}{SNR^{\mathcal{B}}}$ for higher rates) just like we did in Section 8.2.

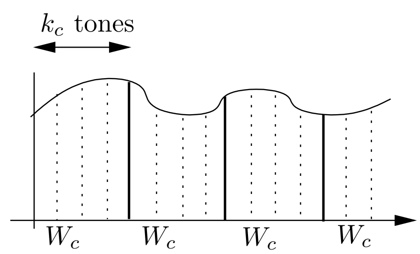


Figure 8.3: An illustration of frequency diversity.

WSSUS model: The argument given above illustrated the idea of frequency diversity but we can have a more precise development for a given channel model. Consider a block time-invariant model which has a finite impulse response with ν taps, *i.e.,*

$$H^{(b)}(D) = h_0^{(b)} + h_1^{(b)} D + ... + h_\nu^{(b)} D^\nu \tag{8.18}$$

Where the superscript (b) denotes the b^{th} transmission block (see also Section 7.2.4). Now, here is where a statistical model of the channel is used. We examine the case where the channel taps $\left\{h_l^{(b)}\right\}$ are i.i.d. for a given block b and are also independent across blocks (this latter model comes from the block time-invariant assumption). Therefore we have i.i.d. $h_l^{(b)} \sim \mathbb{C}\mathcal{N}(0,1), \quad l \in [0, \nu]$ and also i.i.d across blocks.

This model is widely known as the wide-sense stationary uncorrelated scattering model (WSSUS) for a fading ISI Channel. We will use this model to examine the impact of frequency diversity. Note that

$$\begin{aligned}\mathbb{E}\left[H(\omega)H^*(\omega+W_S)\right] &= \mathbb{E}|h_0|^2+\mathbb{E}|h_1|^2e^{jW_S}+...+\mathbb{E}|h_\nu|^2e^{jW_S\nu}\\ &= \sum_{n=0}^{\nu}e^{jnW_S}=\frac{1-e^{j(\nu+1)W_S}}{1-e^{jW_S}}\end{aligned}$$

Therefore for $W_s=\frac{2\pi}{\nu+1}$, we have independence between the frequency responses for the WSSUS model. Therefore the coherence bandwidth $W_c=\frac{W}{\nu+1}$, where W is the signal bandwidth. Thus the larger ν implies a larger frequency diversity for a given transmission bandwidth in the WSSUS model.

8.3.1 OFDM frequency diversity

Now, let us consider using OFDM for the block time-invariant (WSSUS) model given in (8.18). Therefore if we use N carriers for each block we get, (see Section 6.2),

$$Y(n)=H(\frac{2\pi}{N}n)X(n)+Z(n),\quad n=0,...,N-1.$$

First, let us consider repetition coding as was done in Section 8.2.1. Here we do the repetition in frequency, *i.e.,* $X(n)=X(0),\ \forall n$, yielding

$$\begin{aligned}\underbrace{\begin{bmatrix}Y(N-1)\\ .\\ .\\ .\\ Y(0)\end{bmatrix}}_{\mathbf{Y}} &= \underbrace{\begin{bmatrix}H\left(\frac{2\pi}{N}(N-1)\right)\\ .\\ .\\ .\\ H(0)\end{bmatrix}}_{\mathbf{H}}X(0)+\underbrace{\begin{bmatrix}Z(N-1)\\ .\\ .\\ .\\ Z(0)\end{bmatrix}}_{\mathbf{Z}}\\ \mathbf{Y} &= \mathbf{H}X(0)+\mathbf{Z}\end{aligned} \tag{8.19}$$

Notice that (8.19) looks quite similar in form to (8.14).
However we can see from (8.18) that

$$\mathbf{H}=\underbrace{\begin{bmatrix}1 & e^{j\frac{2\pi}{N}(N-1)} & . & . & . & e^{j\frac{2\pi}{N}\nu(N-1)}\\ . & . & & & & .\\ . & . & & & & .\\ . & . & & & & .\\ 1 & 1 & . & . & . & 1\end{bmatrix}}_{\tilde{\mathbf{F}}}\underbrace{\begin{bmatrix}h_0\\ .\\ .\\ .\\ h_\nu\end{bmatrix}}_{\mathbf{h}} \tag{8.20}$$

Note that $\tilde{\mathbf{F}}$ is truncated DFT matrix and is full rank. Using (8.20) in (8.19), we get

$$\mathbf{Y}=\tilde{\mathbf{F}}\mathbf{h}X(0)+\mathbf{Z}$$

A coherent detector for this will erroneously declare $X^{'}(0)$ to be the transmitted symbol instead of $X(0)$ for a *given* realization $\mathbf{h}$ with a probability of

$$\mathbb{P}\left(X(0)\rightarrow X^{'}(0)\mid\mathbf{h}\right)=Q(\sqrt{\frac{[x'(0)-x(0)]^2\mathbf{h}^*\tilde{\mathbf{F}}^*\tilde{\mathbf{F}}\mathbf{h}}{2\sigma_z^2}}) \tag{8.21}$$

where σ_z^2 is the variance of the (white) additive noise $\{z(n)\}$. From (8.20) we see that[1] $\tilde{\mathbf{F}}^*\tilde{\mathbf{F}} = N\mathbf{I}$. Therefore we get,

$$\mathbb{P}\left(X(0) \rightarrow X^{'}(0) \mid \mathbf{h}\right) = Q\left(\sqrt{\frac{[x'(0)-x(0)]^2\mathbf{h}^*\mathbf{h}}{2\sigma_z^2}}\right) \tag{8.22}$$

which is identical in form to (8.15) and hence we get

$$\bar{P}_e(x(0) \rightarrow x'(0)) \doteq \frac{1}{SNR^{\nu+1}},$$

giving us a diversity order of $\nu+1$. Therefore the ISI taps play a role similar to the time diversity conduits given in 8.14.
In a manner similar to Selection 8.1.2, we can develop frequency diversity codes that get a rate larger then $\frac{1}{N+\nu}$ and the analysis of such codes proceeds along similar lines as Section 8.1.2.

8.3.2 Frequency diversity through equalization

One can also exploit the available frequency diversity through the equalizer structures that we have studied in Chapter 5. We will consider the block time-invariant model. First let us examine the matched filter bound which is defined in Section 4.2. This is obtained in our case by sending one symbol every $(\nu+1)$ time instants, and therefore there is no inter-symbol interference. in this case we get

$$SNR_{MFB} = \frac{\mathcal{E}_x}{\sigma_z^2}\left[\int_{\omega} |H(e^{j\omega})|^2 d\omega\right]$$

which is the SNR of a matched filter that collects the energy from all the delayed versions of the transmitted symbol.
However, from the model given in (8.18) we see that

$$H(e^{j\omega}) = \sum_{n=0}^{\nu} h_n e^{j\omega n} = [1, ..., e^{j\omega\nu}]\mathbf{h}$$

and therefore we have $\int |H(e^{j\omega})|^2 d\omega = \mathbf{h}^*\mathbf{h}$, due to the orthogonality of the Fourier transform bases. Therefore the SNR_{MFB} is given by

$$SNR_{MFB} = \frac{\mathcal{E}_x}{\sigma_z^2}||\mathbf{h}||^2$$

and is distributed exactly as in (8.14) and in (8.22) and therefore we expect the detector performance to give an average error probability

$$\bar{P}_{e,MFB} \doteq \frac{1}{SNR^{\nu+1}}$$

and hence the single shot transmission also gives us a diversity order of $\nu+1$ as in OFDM. The main idea illustrated (using simplified models) in Section 8.2.1 and this Section is that frequency diversity through ISI can actually be exploited in a manner similar to time diversity. Actually, by using maximum likelihood sequence detection, if the transmitted sequences satisfy a certain regularity condition, then a diversity order of $\nu+1$ can be obtained.
A natural question is the performance of linear equalizers and whether one can get frequency diversity through them. Let us consider the block time-invariant model, with block size large enough to be able to write the D-domain formulation as

$$Y^{(b)}(D) = H^{(b)}(D)X(D) + Z(D)$$

[1] this can also be seen because $\tilde{\mathbf{F}}$ is a truncated DFT matrix.

where $S_z(D) = \sigma_z^2$. Using the results from Section 5.4 for the MMSE-LE, we have

$$W_{MMSE-LE}(D) = \frac{H^*(D^{-*})\mathcal{E}_x}{H(D)H^*(D^{-*})\mathcal{E}_x + \sigma_z^2}$$

and

$$\begin{aligned} S_{EE}(D) &= \mathcal{E}_x - \frac{H(D)H^*(D^{-*})\mathcal{E}_x^2}{H(D)H^*(D^{-*})\mathcal{E}_x + \sigma_z^2} \\ &= \frac{\sigma_z^2\mathcal{E}_x}{H(D)H^*(D^{-*})\mathcal{E}_x + \sigma_z^2} = \frac{\sigma_z^2}{H(D)H^*(D^{-*}) + \frac{\sigma_z^2}{\mathcal{E}_x}} \end{aligned}$$

Therefore we have

$$SNR_{MMSE-LE} = \frac{\mathcal{E}_x}{\sigma_z^2}\left\{\int_\omega \left[|H(e^{j\omega})|^2 + \frac{\sigma_z^2}{\mathcal{E}_x}\right]^{-1} d\omega\right\}^{-1},$$

which depends on the random variable $\int_\omega \left[|H(e^{j\omega})|^2 + \frac{\sigma_z^2}{\mathcal{E}_x}\right]^{-1} d\omega$.
This computation in general is difficult, however for large SNR one can show that under suitable regularity conditions,

$$SNR_{MMSE-LE} \geq c||\mathbf{h}||^2$$

where c is a constant independent of $\mathbf{h}$. Therefore we obtain

$$\bar{P}_{e,MMSE-LE} \doteq \frac{1}{SNR^{\nu+1}}.$$

We did not go through the details of this argument, but one can make this more precise. Therefore, we see that we can obtain frequency diversity through linear equalization as well.

8.4 Spatial Diversity

In this case one obtains diversity by placing multiple antennas at the transmitter and/or receiver. If the placement is such that the sample "independent" fading in space, one obtains a similar model for diversity as in time diversity. The *crucial* difference is that the spatial diversity is obtained by simultaneously transmitting and receiving symbols and therefore one can potentially increase transmission rate.

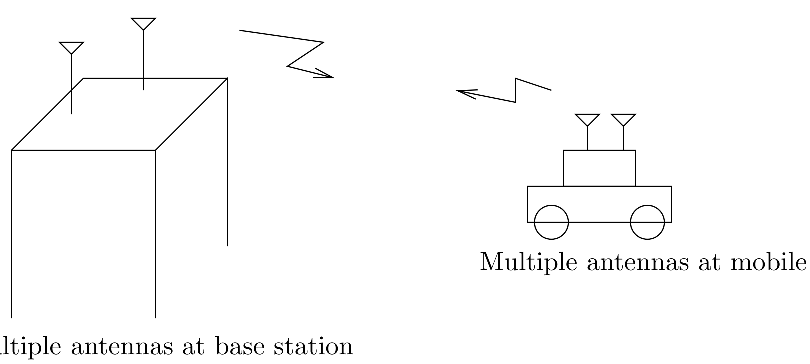


Figure 8.4: Spatial diversity: Multiple transmit and receive antennas.

8.4.1 Receive Diversity

In this case the transmitter is as before, but the reception is done through multiple antennas, *i.e.*,

$$y_l(k) = h_l x(k) + z_l(k),\ l = 1, \dots, \mathcal{D}$$

where $y_l(k)$ is the signal received on l^{th} receive antenna at time instant k, h_l is the fading variable/attenuation on the l^{th} receive antenna, $z_l(k)$ is the additive noise on l^{th} receive antenna and $x(k)$ is the symbol transmitted at time instant k. Let us focus on a *particular* time instant k.

$$\mathbf{y}(k) = \begin{bmatrix} y_1(k) \\ \vdots \\ y_{\mathcal{D}}(k) \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 \\ \vdots \\ h_{\mathcal{D}} \end{bmatrix}}_{\mathbf{h}} x(k) + \underbrace{\begin{bmatrix} z_1(k) \\ \vdots \\ z_{\mathcal{D}}(k) \end{bmatrix}}_{\mathbf{z}(k)}$$

Therefore we get

$$\mathbf{y}(k) = \mathbf{h}x(k) + \mathbf{z}(k) \tag{8.23}$$

This model seems *identical* to equation (8.14) and if the model for fading channel attenuation $\{h(l)\}$ are the same, we would get the *same* performance. Hence, one gets for i.i.d. spatial fading

$$\bar{P}_e(x(k) \to x'(k)) \doteq \frac{1}{(SNR)^{\mathcal{D}}}$$

Note that we get same performance as repetition coding but *without* the rate loss. Another way to observe the performance of receive diversity is as follows. It can easily be shown that a sufficient statistic for (8.23) is

$$\tilde{y}(k) = \mathbf{h}^* \mathbf{y}(k) = \sum_{l=1}^{L} h_l^* y_l(k) = ||\mathbf{h}||^2 x(k) + \mathbf{h}^* \mathbf{z}(k)$$

Hence the receiver combines the signals from each antenna by weighting them according to the channel attenuation, and this is called a **maximal ratio combiner**. This also shows that the SNR at the receiver is $\| h \|^2$ SNR which has a form identical to (8.14) and therefore we get the same average error probability behavior.

Note that multiple antennas imply multiple radio frequency (RF) circuitry associated with each antenna and hence increased cost. Typically multiple antennas are available at the base station and hence one could envisage using them in the above manner to increase performance on "uplink", *i.e.* mobile transmitting to base-station.

8.4.2 Transmit Diversity

If one has multiple antennas at the base-station, a natural question to ask is whether we can utilize them for transmission to a mobile (*i.e.* "downlink" transmission). This has been an active research area for the past decade or so and numerous interesting results have been obtained both for coding and signal processing schemes as well as bounds for fundamental limits of transmission (information theoretic bounds). We will illustrate some of the ideas of the topic through a simple example.

Alamouti code: Consider the scenario with 2 transmit antennas and 1 receive antenna.

$$\begin{aligned} y(k) &= h_1(k) x_1(k) + h_2(k) x_2(k) + z(k) \\ y(k) &= [h_1(k)\ h_2(k)] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + z(k) \end{aligned} \tag{8.24}$$

Now let us examine a code over 2 consecutive instants of time where we assume that the channel is block time-invariant, *i.e.,*

$$h_1(k) = h_1(k+1) = h_1,\ h_2(k) = h_2(k+1) = h_2$$

Moreover, let

$$\begin{aligned} x_1(k) = x_1, &\quad x_1(k+1) = x_2 \\ x_2(k) = -x_2^*, &\quad x_2(k+1) = x_1^* \end{aligned}$$

Then over two consecutive instants of time of (8.24) we get,

$$\underbrace{[y(k),\ y(k+1)]}_{\mathbf{y}} = [h_1\ h_2] \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix} + \underbrace{[z(k),\ z(k+1)]}_{\mathbf{Z}} \tag{8.25}$$

Note that,

$$\begin{aligned} y(k) &= h_1x_1 - h_2x_2^* + z(k) \\ y(k+1) &= h_1x_2 + h_2x_1^* + z(k+1). \end{aligned}$$

Therefore we get,

$$y^*(k+1) = (h_1^*x_2^* + h_2^*x_1) + z^*(k+1)$$

Putting these together we have

$$[y(k),\ y^*(k+1)] = [x_1\ x_2^*] \begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix} + [z(k),\ z^*(k+1)]. \tag{8.26}$$

Therefore, using (8.26) we get,

$$[y(k),\ y^*(k+1)] = \underbrace{[x_1\ x_2^*]}_{\tilde{\mathbf{x}}} \begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix} + \underbrace{[z(k),\ z^*(k+1)]}_{\tilde{\mathbf{z}}} \tag{8.27}$$

Claim 8.4.1. *The matrix* $\begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix}$ *is unitary.*

Proof:

$$\begin{aligned} \begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix} \begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix}^* &= \begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix} \begin{bmatrix} h_1^* & -h_2^* \\ h_2 & h_1 \end{bmatrix} \\ &= \begin{bmatrix} |h_1|^2 + |h_2|^2 & h_1(-h_2^*) + (h_2^*)(h_1) \\ (-h_2)h_1^* + (h_1^*)(h_2) & |h_1|^2 + |h_2|^2 \end{bmatrix} \\ &= \begin{bmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{bmatrix} \\ &= (|h_1|^2 + |h_2|^2)\mathbf{I} \end{aligned}$$

□

Notes:

1. The form of the matrix $\begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix}$ is unitary and this matrix is the linear algebraic representation of the quaternionic group. It is a non-commutative multiplicative group.

2. There are several other forms of the 2×2 matrices of similar natures, *e.g.,*

$$\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}, \begin{bmatrix} -h_1 & h_2 \\ H_2^* & h_1^* \end{bmatrix}, \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix} \text{ etc...}$$

Some of these form multiplicative groups, and others do not.

Now, let us use this property in (8.27)

$$\begin{aligned} [\tilde{y}(k),\ \tilde{y}(k+1)] &= \frac{1}{(|h_1|^2+|h_2|^2)^{1/2}}[y(k),\ y^*(k+1)] \begin{bmatrix} h_1^* & -h_2^* \\ h_2 & h_1 \end{bmatrix} \\ &\overset{(a)}{=} \left(|h_1|^2+|h_2|^2\right)^{1/2}[x_1\ x_2^*] + \frac{1}{(|h_1|^2+|h_2|^2)^{1/2}}[z(k),\ z^*(k+1)] \begin{bmatrix} h_1^* & -h_2^* \\ h_2 & h_1 \end{bmatrix} \end{aligned}$$

where (a) follows due to Claim 8.4.1.
Now, since $\begin{bmatrix} h_1^* & -h_2^* \\ h_2 & h_1 \end{bmatrix}$ is orthogonal, the noise after this transformation remains white, and because of the factor $\frac{1}{(|h_1|^2+|h_2|^2)^{1/2}}$, the transformation is unitary. Therefore the noise is statistically identical to the one before transformation. Hence we get

$$[\tilde{y}(k),\ \tilde{y}(k+1)] = (|h_1|^2+|h_2|^2)^{1/2}[x_1\ x_2^*] + [\tilde{z}(k)\ \tilde{z}(k+1)]$$

which gives,

$$\begin{aligned} \tilde{y}(k) &= (|h_1|^2+|h_2|^2)^{1/2}x_1 + \tilde{z}(k) \\ \tilde{y}(k+1) &= (|h_1|^2+|h_2|^2)^{1/2}x_2^* + \tilde{z}(k+1) \end{aligned}$$

Hence, we can detect x_1 and x_2 independently and we get error probability behavior as

$$\begin{aligned} \mathbb{P}(x_1 \to x_1'|\mathbf{h}) &= Q\left(\frac{||\mathbf{h}||d_{min}}{2\sigma}\right) \\ \mathbb{P}(x_2 \to x_2'|\mathbf{h}) &= Q\left(\frac{||\mathbf{h}||d_{min}}{2\sigma}\right) \end{aligned}$$

Therefore we get the average error probability at high SNR as,

$$\begin{aligned} \bar{P}_e(x_1 \to x_1') &= \mathbb{E}_{\mathbf{h}}\left[\mathbb{P}(x_1 \to x_1'|\mathbf{h})\right] \doteq \frac{1}{SNR^2} \\ \bar{P}_e(x_2 \to x_2') &\doteq \frac{1}{SNR^2} \end{aligned}$$

Therefore, by using this coding scheme at the transmitter, one has utilized maximally two diversity paths which was available.
Notes:

1. The coding scheme achieved similar performance as receive diversity *i.e.* getting error behavior at high SNR as $\bar{P}_e \doteq \frac{1}{SNR^2}$, yielding a diversity order of 2.

2. The decoding scheme shown is actually a maximal likelihood decoder and therefore obtains optimal performance.

3. The decoupling of two *simultaneously* transmitted streams is attained in a simple manner because of the orthogonality of the matrix $\begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix}$.

4. In general if x_1 and x_2 each had a constellation size of $|\mathcal{S}|$, the search space for a maximum likelihood decoder would have been $|\mathcal{S}|^2$. Due to the coding scheme one is able to decouple the problem into 2 problems of size $|\mathcal{S}|$ each, *i.e.* linear decoding complexity of $O(|\mathcal{S}|)$.

5. The transmission rate is still $\log|\mathcal{S}|$, since we transmit $|\mathcal{S}|^2$ information symbols in 2 time instant, *i.e.*,
$$R = \frac{1}{2}\log|\mathcal{S}|^2 = \log|\mathcal{S}|.$$

6. This particularly elegant coding scheme has now become part of next generation wireless network proposals.

7. There are several general principles that can be gleaned from this example. The idea of coding information across multiple transmit antennas (*i.e.* "space") and over multiple transmission instance is called "space-time" coding. There is a deeper theory and a set of design criteria that can be derived for these class of codes which is outside the scope of this chapter.

8.5 Tools for reliable wireless communication

The main idea we learnt in this chapter was that the error probability behavior of fading channels differ significantly from that of time-invariant channels. This is due to the random fluctuations of the channel gain and this led to $\bar{P}_e \doteq \frac{1}{SNR}$. In order to combat this, we needed to use multiple independent instantiations of the channel and code acrosse them. These diversity techniques form the basis tools for increasing reliability over wireless channels. In all the techniques we examined, we obtained an error probability behavior of $\bar{P}_e \doteq \frac{1}{SNR^{\mathcal{D}}}$, by using $\mathcal{D}$ instantiations of randomness. This gives a significant improvement in error probability.
We saw three main diversity mechanisms.

Time diversity: Here we code over time periods longer than the coherence time of the channel and obtained the $\mathcal{D}$ independent channels by this mechanism. Using a single repetition code we could get a diversity order of $\mathcal{D}$. However, we can improve the transmission rate at the cost of diversity order.

Frequency diversity: Here we exploited the "independent" variations in the random frequency response of the ISI channel. Both equalization and multicarrier techniques yielded the maximal diversity order of $(\nu + 1)$, where ν was the order of the WSSUS channel.

Spatial Diversity: We code across multiple transmit antennas and/or receive with multiple antennas. We saw a simple example of coded transmit diversity a.k.a. *space time codes.*

Using diversity is the primary tool for combating channel fading. Many modern techniques combine all three diversity mechanisms for robust performance.

8.6 Problems

Problem 8.1

Assume the channel model
$$y(k) = \alpha(k)x(k) + z(k)$$

Where $y(k)$ is the channel output, $\alpha(k)$ is the flat fading process, $x(k)$ is the channel input and $z(k)$ is the additive white Gaussian noise of zero mean of power σ^2.

1. What is the probability of error for a known value of α?
2. Calculate exactly the average probability of error.

Problem 8.2

Let $\mathbf{y} \sim \mathbb{C}\mathcal{N}(\mathbf{0}, \mathbf{M})$, $\mathbf{Q} = \mathbf{Q}^*$ ($\mathbf{Q}$ need not be positive definite), where $\mathbf{M}, \mathbf{Q} \in \mathbb{C}^{K \times K}$. Prove that the characteristic function $\Phi(\omega)$ of $\mathbf{y}^* \mathbf{Q} \mathbf{y}$ is given by

$$\Phi(\omega) = \mathbb{E}[\exp(j\omega \mathbf{y}^* \mathbf{Q} \mathbf{y})] = \prod_{n=1}^{K} \frac{1}{1 - j\omega\gamma_n},$$

where $\{\gamma_n\}$ are eigen values of the matrix $\mathbf{MQ}$.

Problem 8.3

Consider the transmission of BPSK signal $x = \{\pm a\}$. Let W has distribution $\mathbb{C}\mathcal{N}(0, N_0)$ and H has distribution $\mathbb{C}\mathcal{N}(0, 1)$. Let us assume that we transmit over a flat fading channel with coherent detection

$$Y = HX + W.$$

1. Argue that $r := Re\left(\frac{H^*}{|H|} Y\right)$ is a sufficient statistics for detection of X. Find the distribution of the additive noise in r.
2. Find the exact expression for average (over H) probability of error for part(1) in terms of the $SNR = a^2/N_0$.
3. Assume we transmit over the AWGN channel

   $$Y = X + W.$$

   Find the exact expression for the probability of error in terms of SNR.
4. Compare the average error probability for the flat fading and AWGN channel for $SNR = \{1, 10, 10^2, 10^4, 10^8\}$.

Problem 8.4

Let the channel be given by

$$\underbrace{\begin{bmatrix} y(D-1) \\ \vdots \\ y(0) \end{bmatrix}}_{\mathbf{Y}} = \underbrace{\begin{bmatrix} h(D-1) \\ \vdots \\ h(0) \end{bmatrix}}_{\mathbf{h}} x(0) + \underbrace{\begin{bmatrix} z(D-1) \\ \vdots \\ z(0) \end{bmatrix}}_{\mathbf{z}}$$

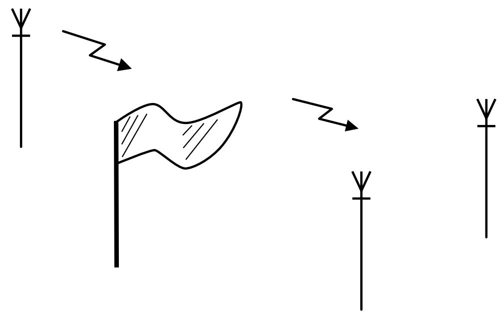
Assume that $\{h(l)\}$ is perfectly known at the receiver and therefore we can form a coherent detector. If $x(0)$ was the transmitted symbol, then the probability that $x(0)$ is mistaken for $x'(0)$ is

$$\begin{aligned} \mathbb{P}x(0) \to x'(0)|\mathbf{h} &= \mathbb{P}||\mathbf{y} - \mathbf{h}x'(0)||^2 < ||\mathbf{y} - \mathbf{h}x(0)||^2|\mathbf{h} \\ &= Q\left(\frac{||\mathbf{h}||d_{min}}{2\sigma}\right), \end{aligned}$$

where $||\mathbf{h}||^2 = \sum_{\mathbf{l=0}}^{\mathbf{L-1}} |\mathbf{h}(\mathbf{l})|^2$.
Suppose that $\mathbf{h} \sim \mathbb{C}\eta(0, \mathbf{K_h})$, where $\mathbf{K_h}$ is not necessarily diagonal.

1. Find the bound on average probability of error assuming high SNR.
2. Assume that the matrix K_h is positive definite. Find the diversity order.

Problem 8.5



Suppose there is a transmitter which is sending signals to be received by two receive antennas. However due to a strange and unfortunate coincidence there is a flag fluttering in the wind quite close to one of the receive antennas and sometimes completely blocks the received signal.
In the absence of the flag, the received signal is given by a flat fading model, (discrete time model as done in class).

$$\mathbf{Y}_k = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} h_1(k) \\ h_2(k) \end{bmatrix} x(k) + \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \tag{8.28}$$

where $y_1(k)$, $y_2(k)$ are the received signals on first and second receive antennas respectively, $x(k)$ is the transmitted signal and $h_1(k)$, $h_2(k)$ are respectively the fading attenuation from the transmitter to the first and second receive antennas. Assume that $x(k)$ is binary, i.e. $x(k) \in \{-\sqrt{\mathcal{E}_x}, \sqrt{\mathcal{E}_x}\}$. The additive noise $z_1(k)$, $z_2(k)$ are assumed to be independent circularly symmetric complex Gaussian with variance (each) of σ^2. Assume that $h_1(k)$, $h_2(k)$ are i.i.d complex Gaussian $\mathbb{C}\eta(0, 1)$.

1. Over several transmission blocks, compute the upper bound to the error probability and comment about the behavior of the error probability with respect to SNR for high SNR. *Hint*: Use the fact that $Q(x) \leq e^{-x^2/2}$.
2. Now let us consider the presence of fluttering flag which could potentially block only the second receive antenna. The model given in (8.28) now changes to:

$$\mathbf{Y}_k = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} h_1(k) \\ \mathcal{F}_k h_2(k) \end{bmatrix} x(k) + \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}$$

   where:

$$\mathcal{F}_k = \begin{cases} 1 & \text{if there is no obstruction from the flag} \\ 0 & \text{if flag obstructs} \end{cases}$$

   Suppose due to the random fluttering, the flag blocks a fraction q of the transmissions, i.e for a fraction q of the transmission, one receives only the signal from the first antenna.

   Conditioned on $\mathcal{F}$, write down the error probabilities, i.e find and expression for $P_e(x \to x'|\mathcal{F})$ and compute its upper bound (see hint in (1)).

3. Find the overall error probability in the presence of the fluttering. How does the error probability behave at high SNR, i.e what diversity order does one obtain.

Hint: If the error probability behaves as $\frac{1}{SNR^D}$ at high SNR, the diversity order is D.

Problem 8.6

We have studied time diversity with repetition coding for coherent detection. Let number of diversity paths $D = 2$. A repetition coding repeats the BPSK symbol $u = \pm a$ over both the diversity paths. In this problem we study another coding scheme by which we need not sacrifice the rate like in the repetition coding. Consider instead a coding scheme

$$\mathbf{x} = \mathbf{R}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

where $u_1, u_2 \in \{\pm a\}$ and the matrix $\mathbf{R}$ is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

$\mathbf{R}$ is a rotation matrix (for some $\theta \in (0, 2\pi)$). Now the received signal is given by:

$$y_l = h_l x_l + z_l, \quad l = 1, 2.$$

where $\{h_l\}$ are i.i.d and has the distribution $\mathbb{C}N(0, 1)$.

1. Suppose that we transmit $\mathbf{x_A} = \mathbf{R}\begin{bmatrix} a \\ a \end{bmatrix}$ and use $Re\left(\frac{h_l^* y_l}{|h_l|}\right)$ as the sufficient statistics. Use the union bound to upper bound the probability of error.

2. What should be the condition on θ so that we obtain a diversity order of 2?

Problem 8.7

We have studied the performance of the Alamouti scheme in a system with two transmit and one receive antenna. Suppose now we have an additional receive antenna. Assume that the channel is block time invariant for 2-consecutive transmissions. Let the input be BPSK i.e. $x_i \in \{\pm a\}$ and we assume coherent detection.

1. Denote the received vector (with appropriate conjugation of the components as required in the Alamouti scheme) at the two receiving antennas over two consecutive transmissions by $\mathbf{y} \in \mathbb{C}^4$. Let the channel model be $\mathbf{y} = \mathbf{H}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{z}$, where $\mathbf{z} \in \mathbb{C}^4$ and has the distribution $\mathbb{C}N(0, N_0 I_4)$. Write down explicitly $\mathbf{y}, \mathbf{H}$.

2. Show that the columns of $\mathbf{H}$ are orthogonal. Using this show that projection of $\mathbf{y}$ on the normalized column vectors of $\mathbf{H}$ gives sufficient statistics for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

3. Write down the ML decision rule and find the upper bound on average probability of error. Find the diversity gain for this scheme.

Problem 8.8

1. Show that $1_{\{x-x_0\geq 0\}} \leq e^{\gamma(x-x_0)}$. Using this, prove $Pr(x \geq x_0) \leq \mathbb{E}[e^{\gamma(x-x_0)}]$ for any $\gamma > 0$.
2. Use this to prove an upper bound for $Pr(||y-hx|| \geq ||y-hx'|| \,|h, x)$ where $y = hx + z$. (here you would get the exponential bound). Optimize w.r.t γ to get the $e^{-h*(x-x')(x-x')^*h/4\sigma 2}$ bound.
3. From here, use the distribution of h to get the 1/SNR type of behaviour.

Problem 8.9

Show that the quaternionic group is a non-commutative multiplicative group.

Appendices for Chapter 8

8.A Exact Calculations of Coherent Error Probability

First let us examine the probability distribution of $|h|$.
Since,

$$|h|^2 = [\mathrm{Re}(h)]^2 + [\mathrm{Im}(h)]^2$$

where $\mathrm{Re}(h) \sim \mathcal{N}(0, \frac{1}{2})$ and $\mathrm{Im}(h) \sim \mathcal{N}(0, \frac{1}{2})$ as h is a circular symmetric Complex Gaussian random variable. It can be therefore seen[2] that if $u = |h|^2$,

$$f_V(u) = e^{-u} \tag{8.29}$$

Now, using (8.29) in (8.4) we get

$$\bar{P}_e = \int_{u=0}^{\infty} e^{-u} Q(a\sqrt{u})du$$

where we have defined $a = \frac{d_{min}}{2\sigma_z}$. Therefore we can write

$$\begin{aligned}
\bar{P}_e &= \int_{u=0}^{\infty} e^{-u}\left[\int_{\sqrt{2}a\sqrt{u}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv\right] du = \int_{v=0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \left[\int_0^{\frac{v^2}{a^2}} e^{-u} du\right] dv \\
&= \int_{v=0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} (1 - e^{-\frac{v^2}{a^2}}) dv = \frac{1}{2} - \int_{v=0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2\left[\frac{1}{2}+\frac{1}{a^2}\right]} dv \\
&= \frac{1}{2} - \gamma \int_{v=0}^{\infty} \frac{1}{\sqrt{2\pi}\gamma} e^{-\frac{v^2}{2\gamma^2}} dv = \frac{1}{2}\left[1-\gamma\right]
\end{aligned}$$

where $\gamma^2 = \frac{1}{1+\frac{2}{a^2}}$.
Hence

$$\bar{P}_e = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1+(\frac{2}{a^2})}}\right] = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1+\frac{8\sigma_z^2}{d_{min}^2}}}\right]. \tag{8.30}$$

[2] Since $\mathrm{Re}(h)$ and $\mathrm{Im}(h)$ are i.i.d. real Gaussian random variables, $[\mathrm{Re}(h)]^2$ and $[\mathrm{Im}(h)]^2$ are also i.i.d. with density function $P_V(v) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{v}{2\sigma^2}}$, where $v = \mathrm{Re(h)}$ or $v = \mathrm{Im}(h)$, and $\sigma^2 = \mathbb{E}[\mathrm{Re}(h)]^2 = \mathbb{E}[\mathrm{Im}(h)]^2 = \frac{1}{2}$. Hence the characteristic function of $[\mathrm{Re}(h)]^2$ is $\mathbb{E}[e^{j\omega[\mathrm{Re}(h)]^2}] = \frac{1}{(1-j2\omega\sigma^2)^{\frac{1}{2}}}$. This allows us to write the characteristic function of $[\mathrm{Re}(h)]^2 + [\mathrm{Im}(h)]^2$ as $\frac{1}{1-j2\omega\sigma^2}$, and hence the inverse Fourier transform of this yields (8.29).

8.B Non-coherent detection: fast time variation

In order to illustrate rapid time variation we consider extremely small block sizes, *i.e.,* blocks of size one, which yields

$$y(k) = h(k)x(k) + z(k)$$

where $\{h(k)\}$ are Gaussian with a given covariance. If we want to do maximum-likelihood detection, we form, $\mathbb{P}_{Y|X}(y|x)$. Note that here $h(k)$ is also a source of randomness and hence for each k we have,

$$\mathbb{P}_{Y|X}(y|x) \sim \mathbb{C}\mathcal{N}\left(0, \left\{\mathbb{E}[|h|^2]\right\}|x|^2 + \sigma_z^2\right) \tag{8.31}$$

Therefore as one would expect from this relationship the maximum likelihood criterion takes a form that is substantially different from what we were used to in Section 8.1.1.
Let us form the detector for a sequence of transmitted symbols. Furthermore, let us assume that for the short interval of the transmission the channel has a given statistical variation. With no loss in generality let us assume that we transmit a block of K symbols from time 0 to $K-1$

$$\begin{aligned}
\mathbf{y} = \begin{bmatrix} y(K-1) \\ \vdots \\ y(0) \end{bmatrix} &= \underbrace{\begin{bmatrix} h(K-1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & h(0) \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x(K-1) \\ \vdots \\ x(0) \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} z(K-1) \\ \vdots \\ z(0) \end{bmatrix}}_{\mathbf{Z}} \\
&= \underbrace{\begin{bmatrix} x(K-1) & & \\ & \ddots & \\ & & x(0) \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} h(K-1) \\ \vdots \\ h(0) \end{bmatrix}}_{\mathbf{h}} + \mathbf{z}
\end{aligned} \tag{8.32}$$

For a given $\mathcal{X}$, using the same idea as in (8.31), we get for (8.32)

$$\mathbf{y} \sim \mathbb{C}\mathcal{N}(0, \mathcal{X}\mathbb{E}\left[\mathbf{h}\mathbf{h}^*\right]\mathcal{X}^* + \sigma_z^2\mathbf{I}_K)$$

where $\mathbf{R}_h = \mathbb{E}\left[\mathbf{h}\mathbf{h}^*\right]$ depends on the time-correlation of the (fading) channel random process. For the maximum likelihood rule, we have

$$\mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = \frac{1}{\pi|\mathcal{X}\mathbf{R}_h\mathcal{X}^* + \sigma_z^2\mathbf{I}_K|} \exp\left(-\mathbf{y}^*[\mathcal{X}\mathbf{R}_h\mathcal{X}^* + \sigma_z^2\mathbf{I}_K]^{-1}\mathbf{y}\right)$$

Therefore the log-likelihood is,

$$\log \mathbb{P}_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \quad = \quad -\log\pi - \log|\mathcal{X}\mathbf{R}_h\mathcal{X}^* + \sigma_z^2\mathbf{I}_K| - \mathbf{y}^*[\mathcal{X}\mathbf{R}_h\mathcal{X}^* + \sigma_z^2\mathbf{I}_K]^{-1}\mathbf{y}$$

Consider the important special case when $|x(i)|^2 = \mathcal{E}_x$, *i.e.* we have a constant modulus constellation (*e.g.,* PSK).
Hence if $|x(i)|^2 = \mathcal{E}_x$ then $\mathcal{X}\mathcal{X}^* = \mathcal{E}_x\mathbf{I}_K$, hence

$$\begin{aligned}
|\mathcal{X}\mathbf{R}_h\mathcal{X}^* + \sigma_z^2\mathbf{I}_K| &= |\mathcal{X}\mathbf{R}_h\mathcal{X}^* + \frac{\sigma_z^2}{\mathcal{E}_x}\mathcal{X}\mathcal{X}^*| \\
&= |\mathcal{X}(\mathbf{R}_h + \frac{\sigma_z^2}{\mathcal{E}_x}\mathbf{I}_K)\mathcal{X}^*| \\
&\overset{(a)}{=} \left|\left[\mathbf{R}_h + \frac{\sigma_z^2}{\mathcal{E}_x}\mathbf{I}_K\right]\mathcal{E}_x\right|
\end{aligned}$$

where (a) follows because $|x(i)|^2 = \mathcal{E}_x \ \forall i$.
Hence we get the maximum likelihood rule as,

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \left\{ -\mathbf{y}^* \left[\sigma_z^2 \mathbf{I}_K + \mathcal{X}\mathbf{R}_h\mathcal{X}^* \right]^{-1} \mathbf{y} \right\} \tag{8.33}$$

Now by the matrix inversion lemma, Lemma 5.7.1 we have

$$\left[\sigma_z^2 \mathbf{I}_K + \mathcal{X}\mathbf{R}_h\mathcal{X}^* \right]^{-1} = \frac{1}{\sigma_z^2}\mathbf{I}_K - \frac{1}{\sigma_z^2}\mathcal{X}\left(\mathbf{R}_h^{-1} + \frac{\mathcal{X}\mathcal{X}^*}{\sigma_z^2} \right)^{-1} \frac{1}{\sigma_z^2}\mathcal{X}^*$$

Therefore using this in (8.33) we get

$$\begin{aligned} \hat{\mathbf{x}} &= \arg\max_{\mathbf{x}} \left\{ \mathbf{y}^*\mathcal{X} \left[\mathbf{R}_h^{-1} + \frac{\mathcal{X}\mathcal{X}^*}{\sigma_z^2} \right]^{-1} \mathcal{X}^*\mathbf{y} \right\} \\ &= \arg\max_{\mathbf{x}} \left\{ \mathbf{y}^*\mathcal{X} \left[\mathbf{R}_h^{-1} + \frac{\mathcal{E}_x}{\sigma_z^2}\mathbf{I}_K \right]^{-1} \mathcal{X}^*\mathbf{y} \right\} \end{aligned} \tag{8.34}$$

This projects the received signal onto the space spanned by the codeword and a weighted norm is taken. This idea of projecting the received signal onto the space spanned by the codewords is actually a useful and important idea for non-coherent detection.
We can recover from this the result in Section 8.1.2. For the block time-invariant model considered there we have,

$$\mathbf{R}_h = \mathbf{1}\mathbf{1}^t$$

where $\mathbf{1}^t = [1, \ldots, 1]$. Then, the detection criterion given in (8.33) becomes,

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \left\{ -\mathbf{y}^* \left[\sigma_z^2 \mathbf{I}_K + \mathcal{X}\mathbf{1}\mathbf{1}^t\mathcal{X}^* \right]^{-1} \mathbf{y} \right\} \tag{8.35}$$

Now,

$$\mathcal{X}\mathbf{1} = \begin{bmatrix} x(N-1) \\ \vdots \\ x(0) \end{bmatrix} = \mathbf{x}.$$

Therefore using this in (8.35) we have

$$\left[\sigma_z^2 \mathbf{I}_K + \mathbf{x}\mathbf{x}^* \right]^{-1} = \frac{1}{\sigma_z^2}\mathbf{I}_K - \frac{1}{\sigma_z^2}\mathbf{x}\left(1 + \frac{\mathbf{x}^*\mathbf{x}}{\sigma_z^2} \right)^{-1} \frac{1}{\sigma_z^2}\mathbf{x}^*.$$

Hence the criterion becomes,

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \left\{ \frac{\mathbf{y}^*\mathbf{x}\mathbf{x}^*\mathbf{y}}{\left(1 + \frac{\mathcal{E}_x K}{\sigma_z^2}\right)(\sigma_z^2)^2} \right\}.$$

If $|x(i)|^2 = \mathcal{E}_x$ we get $\mathbf{x}^*\mathbf{x} = N\mathcal{E}_x$, therefore we have

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \left\{ |\mathbf{x}^*\mathbf{y}|^2 \frac{1}{(\sigma_z^2)^2 \left(1 + \frac{\mathcal{E}_x K}{\sigma_z^2}\right)} \right\} = \arg\max_{\mathbf{x}} |\mathbf{x}^*\mathbf{y}|^2,$$

which is the same criterion derived in Section 8.1.2.

8.C Error probability for non-coherent detector

Let us consider the error probability for the non-coherent detector given in 8.1.2. For such a detector, if $\mathbf{x}$ was sent we detect it to be $\hat{\mathbf{x}} \neq \mathbf{x}$ if

$$\begin{aligned}
\mathbb{P}\left\{\mathbf{x} \to \hat{\mathbf{x}}\right\} &= \mathbb{P}\left\{\frac{1}{(\sigma_z^2)^2}\left|\mathbf{x}^*\mathbf{y}\right|^2 < \frac{1}{(\sigma_z^2)^2}\left|\hat{\mathbf{x}}^*\mathbf{y}\right|^2\right\} \\
&= \mathbb{P}\left\{\frac{1}{(\sigma_z^2)^2}\mathbf{y}^*\left[\hat{\mathbf{x}}\hat{\mathbf{x}}^* - \mathbf{x}\mathbf{x}^*\right]\mathbf{y} > 0\right\} \qquad (8.36)
\end{aligned}$$

Let us define

$$\mathbf{Q} = \frac{1}{(\sigma_z^2)^2}\left\{\hat{\mathbf{x}}\hat{\mathbf{x}}^* - \mathbf{x}\mathbf{x}^*\right\}$$

and notice that since $\mathbf{x}$ was transmitted

$$\mathbf{y} \sim \mathbb{C}\mathcal{N}(0, \mathbf{x}\mathbf{x}^* + \sigma_z^2\mathbf{I}_K).$$

So we have the error probability of a Gaussian quadratic form where

$$\mathbb{P}\{\mathbf{x} \to \hat{\mathbf{x}}\} = \mathbb{P}\{\mathbf{y}^*\mathbf{Q}\mathbf{y} > 0\}.$$

and $\mathbf{Q} = \mathbf{Q}^*$, but need not be positive-semidefinite, *i.e.,* could have negative eigenvalues.

Lemma 8.C.1. *If* $\mathbf{y} \sim \mathbb{C}\mathcal{N}(0, \mathbf{M})$ *and* $\mathbf{Q} = \mathbf{Q}^*$, *then*

$$\mathbb{P}\left\{\mathbf{y}^*Q\mathbf{y} > 0\right\} = \sum_{\gamma_i < 0}\prod_{n=1}^{K}\left(\frac{1}{1 - \frac{\gamma_n}{\gamma_i}}\right)$$

where $\mathbf{M} \in \mathbb{C}^{K\times K}$ *and* $\{\gamma_n\}$ *are the eigenvalues of* $\mathbf{MQ}$.

Proof: The characteristic function of $\mathbf{y}^*\mathbf{Q}\mathbf{y}$ is given by

$$\Phi(\omega) = \mathbb{E}\left\{exp\left(j\omega\mathbf{y}^*\mathbf{Q}\mathbf{y}\right)\right\} = \prod_{n=1}^{K}\frac{1}{1 - j\omega\gamma_n}$$

where $\{\gamma_n\}$ are the eigenvalues of the matrix $\mathbf{MQ}$.
Therefore, the error probability is

$$\begin{aligned}
\mathbb{P}\left\{\mathbf{x} \to \hat{\mathbf{x}}\right\} &= \mathbb{P}\left\{\mathbf{y}^*\mathbf{Q}\mathbf{y} > 0\right\} \qquad (8.37) \\
&= \int_0^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)e^{-j\omega u}d\omega\right]du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)\left[\int_0^{\infty}e^{-j\omega u}du\right]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)\left[\pi\delta(\omega) - \frac{1}{j\omega}\right]d\omega \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\Phi(\omega)\delta(\omega)d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega}\prod_{n=1}^{K}\frac{1}{(1 - j\omega\gamma_n)}d\omega \\
&= \frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega}\prod_{n=1}^{K}\left(\frac{1}{1 - j\omega\gamma_n}\right)d\omega
\end{aligned}$$

In order to evaluate this integral we use the residue theorem which is stated for convenience below.

Theorem 8.C.1. *(**Residue theorem**) Let f(z) be a function which is analytic inside a simple closed path $\mathcal{C}$ and on $\mathcal{C}$, except for a finitely many singular points $a_1, \dots, a_m$ inside $\mathcal{C}$. Then*

$$\int_{\mathcal{C}} f(z)dz = 2\pi j \sum_{i=1}^{m} \texttt{Res}\left\{f(z)\right\}_{z=a_i}$$

where for a simple pole a_i of $f(z)$, $\texttt{Res}\left\{f(z)\right\}_{z=a_i} = (z-a_i)f(z)|_{z=a_i}$.

□

In order to use this result we use the integral path on the upper half of the $\mathbb{C}$ plane in (8.38). Notice that,

$$\frac{1}{1-j\omega\gamma_n} = \frac{j\gamma_n^{-1}}{j\gamma_n^{-1}+\omega}$$

Therefore using this and the residue theorem 8.C.1 we get,

$$\frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{j\omega}\prod_{n=1}^{K}\left(\frac{1}{1-j\omega\gamma_n}\right)d\omega \quad = \quad \frac{1}{2} - \left[\frac{1}{2} - \sum_{\gamma_i<0}\prod_{n=1, n\neq i}^{K}\left(\frac{1}{1-\frac{\gamma_n}{\gamma_i}}\right)\right]$$

Therefore, we get the stated result,

$$\mathbb{P}\left\{\mathbf{x}\to\hat{\mathbf{x}}\right\} \quad = \quad \sum_{\gamma_i<0}\prod_{n=1}^{K}\left(\frac{1}{1-\frac{\gamma_n}{\gamma_i}}\right).$$

□

Let $u_1 = \frac{1}{\sigma_z^2}\hat{\mathbf{x}}^*\mathbf{y}$, $u_2 = \frac{1}{\sigma_z^2}\mathbf{x}^*\mathbf{y}$. Clearly, u_1 and u_2 are scalar complex Gaussian random variables.

$$\begin{aligned}
\mathbb{E}|u_1|^2 &= \frac{1}{(\sigma_z^2)^2}\hat{\mathbf{x}}^*[\sigma_z^2\mathbf{I}+\mathbf{x}\mathbf{x}^*]\hat{\mathbf{x}} = K\rho + \frac{1}{(\sigma_z^2)^2}|\hat{\mathbf{x}}^*\mathbf{x}|^2 = K\rho + |\rho K\beta|^2 \\
\mathbb{E}|u_2|^2 &= \frac{1}{(\sigma_z^2)^2}\left\{K\mathcal{E}_x\sigma_z^2 + (K\mathcal{E}_x)^2\right\} = K\rho[1+K\rho] \\
\mathbb{E}u_1u_2^* &= \frac{1}{(\sigma_z^2)^2}\hat{\mathbf{x}}^*[\sigma_z^2\mathbf{I}+\mathbf{x}\mathbf{x}^*]\mathbf{x} = K\rho\beta + K\rho\beta K\rho = (K\rho\beta)[1+\rho K]
\end{aligned}$$

where we have defined $\hat{\mathbf{x}}^*\mathbf{x} = \mathcal{E}_x K\beta$ and $\rho = \frac{\mathcal{E}_x}{\sigma_z^2}$. Using this in (8.36) we get,

$$\begin{aligned}
\mathbb{P}\{\mathbf{x}\to\hat{\mathbf{x}}\} &= \mathbb{P}\left\{|u_1|^2 - |u_2|^2 > 0\right\} \\
&= \mathbb{P}\left\{[u_1^*, u_2^*]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} > 0\right\}.
\end{aligned}$$

Now let us use this in Lemma 8.C.1, we get

$$\mathbf{M} = \begin{bmatrix} K\rho(1+K\rho|\beta|^2), & (K\rho\beta)[1+\rho K] \\ (K\rho\beta^*)[1+\rho K], & K\rho[1+K\rho] \end{bmatrix}$$

Also we have

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Clearly, the conditions of of Lemma 8.C.1 are satisfied and so we need the eigenvalues of $\mathbf{MQ}$. Now we have,

$$\begin{aligned}
\mathbf{MQ} &= \begin{bmatrix} K\rho(1+K\rho|\beta|^2), & -(K\rho\beta)[1+\rho K] \\ (K\rho\beta^*)[1+\rho K], & -K\rho[1+K\rho] \end{bmatrix} \\
&= K\rho[1+\rho K]\begin{bmatrix} \frac{1+K\rho|\beta|^2}{[1+\rho K]}, & -\beta \\ \beta^*, & -1 \end{bmatrix}
\end{aligned}$$

Therefore the eigenvalues of $\mathbf{MQ}$ are,

$$\begin{aligned}
\gamma_1 &= K\rho[1+\rho K]\lambda_1 \\
\gamma_2 &= K\rho[1+\rho K]\lambda_2
\end{aligned}$$

where λ_1, λ_2 are the eigenvalues of

$$\begin{bmatrix} \frac{1+K\rho|\beta|^2}{[1+\rho K]}, & -\beta \\ \beta^*, & -1 \end{bmatrix}$$

and hence for $\Gamma = \frac{1+K\rho|\beta|^2}{1+K\rho}$, we have

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\left\{(\Gamma-1) - \sqrt{(\Gamma-1)^2 + 4(\Gamma-|\beta|^2)}\right\} \\
\lambda_2 &= \frac{1}{2}\left\{(\Gamma-1) + \sqrt{(\Gamma-1)^2 + 4(\Gamma-|\beta|^2)}\right\}
\end{aligned}$$

and observing that $1-\Gamma = \frac{K\rho(1-|\beta|^2)}{1+K\rho}$ and $\Gamma - |\beta|^2 = \frac{1-|\beta|^2}{1+K\rho}$, we get,

$$\begin{aligned}
\lambda_1 &= \frac{1}{2(1+K\rho)}\left\{-K\rho(1-|\beta|^2) - \sqrt{\{K\rho(1-|\beta|^2)\}^2 + 4(1-|\beta|^2)(1+K\rho)}\right\} < 0 \\
\lambda_2 &= \frac{1}{2(1+K\rho)}\left\{-K\rho(1-|\beta|^2) + \sqrt{\{K\rho(1-|\beta|^2)\}^2 + 4(1-|\beta|^2)(1+K\rho)}\right\} > 0
\end{aligned}$$

Hence we have,

$$\begin{aligned}
\gamma_1 &= \frac{(K\rho)^2(1-|\beta|^2)}{2}\left\{-1 - \sqrt{1 + \frac{4(1+K\rho)}{(K\rho)^2(1-|\beta|^2)}}\right\} < 0 \\
\gamma_2 &= \frac{(K\rho)^2(1-|\beta|^2)}{2}\left\{-1 + \sqrt{1 + \frac{4(1+K\rho)}{(K\rho)^2(1-|\beta|^2)}}\right\} > 0
\end{aligned}$$

Using Lemma 8.C.1, we get

$$\mathbb{P}\{\mathbf{x} \to \hat{\mathbf{x}}\} = \frac{1}{1 - \frac{\gamma_2}{\gamma_1}} = \frac{-\gamma_1}{\gamma_2 - \gamma_1}$$

Hence we get at high SNR,

$$\mathbb{P}\{\mathbf{x} \to \hat{\mathbf{x}}\} \doteq \frac{1}{SNR}.$$

Therefore the high SNR behavior of the error probability is like $\frac{1}{SNR}$, which is like the coherent case.

Chapter 9

Multi-user communication

The wireless propagation medium is (almost by definition!) shared by many simultaneously transmitting users. For example one has seen that several people use their mobile phones simultaneously in a small geographical area. Therefore, these users' signals would overlap and interfere with one another. In this chapter we study basic communication topologies which arise in wireless networks. We study how multiple users can share and access common resources. After studying these in Section 9.1, we focus our attention on multiple access channels in Section 9.3 and 9.4. In Section 9.2 we will study different access techniques for multiuser channels. In Section 9.3, we set up the model and sufficient statistics for the multiple access problem. In Section 9.4 we study multi-user detectors that naturally generalize some of the ideas from Chapter 4 and 5.

9.1 Communication topologies

In order to build a multiuser wireless network the first question is how to share the common "channel" resources among the different users.

To address this question let us first examine the various configurations that can arise in multiuser networks. There are two categories of configurations. One is hierarchical where there is a central access point and users communicate to the access point. The other configuration could be where the users are allowed to communicate directly to one another perhaps using other users as relays. Such configurations are called ad hoc networks and we briefly talk about them in Section 9.1.2.

9.1.1 Hierarchical networks

In most current wireless systems there are two main hierarchical configurations that are used.

Broadcast Base-station to user (one-to-many) broadcast or downlink communication.

Multiple access Users to base-station (many-to-one) multiple access or uplink communication.

These configuration are shown in figures 9.1 and 9.2. In the "downlink" broadcast communication, the (server) base-station communicates to the users. This can be the case where the base-station sends information like web-downloads or voice information to the mobile user.

The "uplink" multiple access communication is exactly the reverse of this, *i.e.*, many users send information (to be transferred) to the base-station (server). This could occur when the users send a request, or upload data or are sending a voice message/conversation to the base-station intending it to be forwarded to its final destination.

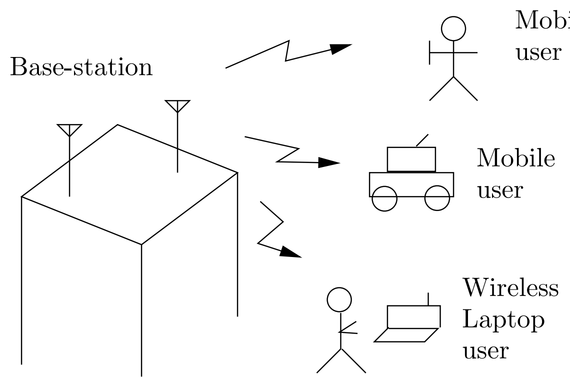


Figure 9.1: Base-station to user (one-to-many) broadcast/downlink communication.

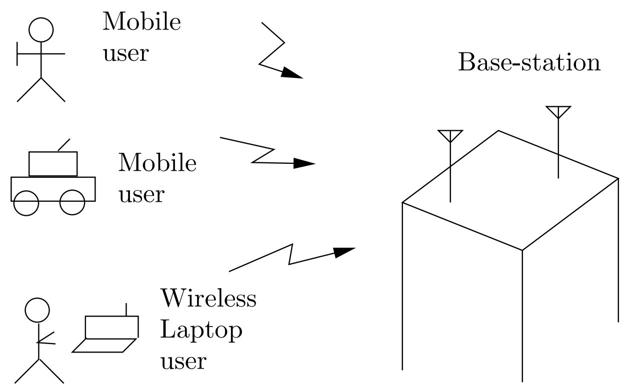


Figure 9.2: Users to base-station (many-to-one) multiple access/uplink communication.

In both cases, the users receive/transmit signals "simultaneously". Therefore, in order to help distinguish the signals from the different users, typically the channel resources (time, frequency) are divided among the users. This idea leads to the main "channel access" techniques used in current systems. These access schemes are called Time-Division Multiple Access (TDMA), Frequency-Division Multiple Access (FDMA) and Code-division Multiple Access (CDMA).

9.1.2 Ad hoc wireless networks

In contrast to hierarchical networks where we have a base-station serving a cluster of mobiles (see Figures 9.1 and 9.2) another configuration is where there is no central infrastructure. These networks variously called ad hoc networks, self organized networks, mesh networks etc. They are primarily used in military applications where there was a need to have an architecture without clear vulnerabilities. In an ad hoc network, the mobiles can form any configuration depending on the communication conditions and needs. The most distinct feature of such network is the use of relays to transmit information from one node to another. This is illustrated in figure 9.3.

For example, if source node $\mathcal{S}$ wants to transmit to destination $\mathcal{T}$ it can transmit through several alternate routes depending on the transmission conditions. For example in Figure 9.3, route 1, relays the information through relay node A, and route 2 sends it through nodes B and C.

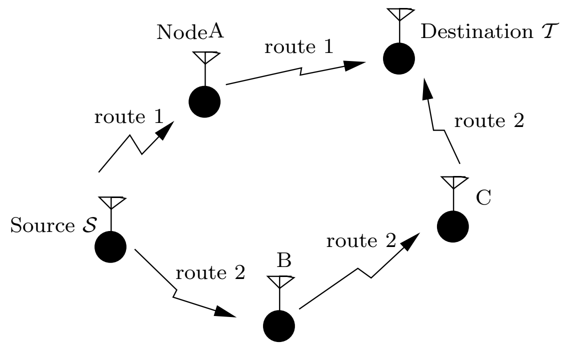


Figure 9.3: Adhoc or mesh wireless networks.

The advantage of ad hoc networks in that they are reconfigurable, *i.e.,* if any nodes fails or the communication conditions (*e.g.* fading) changes, then the transmission routes can also change. They can also take advantage of varying communication needs, for example the node $\mathcal{S}$ might be the only one who needs to send information and therefore the network resources can be manoeuvered for this purpose. This flexibility makes such an architecture attractive. However, this functionality also comes at a cost of overhead in setting up routes, locating nodes and doing so in a distributed manner since there is no "central authority". Therefore, this configuration/architecture is yet to be implemented in commercial systems through the 802.11 wireless LAN standard does have this as an option (though rarely used!). Ad hoc networks also have some interesting capacity scaling properties, but this is a topic beyond the scope of this chapter.

9.2 Access techniques

Access techniques refer to methods by which users establish communication to each other through the shared medium. This can be studied both in the context of hierarchical networks as well as ad hoc networks. We will only discuss deterministic access techniques here. However, there are random access techniques which utilize the resources adaptively depending on the needs.

9.2.1 Time Division Multiple Access (TDMA)

In this, the transmission time is divided into slots and each user is assigned a distinct time slot therefore getting exclusive use of the channel during this period and there is no interference from other users in the same "cell". This is shown in figure 9.4

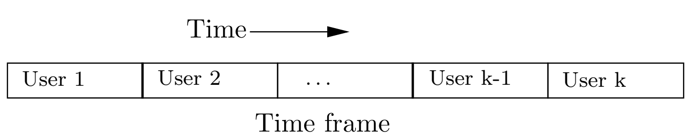


Figure 9.4: Time division Multiple Access.

All users must have a synchronized clock to know exactly when their transmission begins. Since users are geographically separated, they also need to account for (small) propagation delays that might occur. Typically the time-periods are equal in size and therefore the resources are equally divided among the users. Some recent proposals use unequals slots or assign more than one slot to a user in order to give differentiated services to the users. TDMA is implemented in several current wireless systems, both in the USA, Europe and in Japan. These systems are used for mobile telephony are called IS-136, GSM etc.

9.2.2 Frequency Division Multiple Access (FDMA)

This is the frequency dual of TDMA, where instead of time, frequency is broken up into slots and users are assigned exclusive use of the small piece of the frequency spectrum. Again, this is used for both uplink and downlink. Most commercial systems use both TDMA and FDMA to divide the resources. The frequency is divided and then each part could be further divided using TDMA. A more modern method for FDMA involves use of OFDM which we have seen in Section 6.2. In this, the users transmit on a wide bandwidth and they are assigned "carriers" or "tones" of the OFDM scheme. They transmit (or listen) only on those tones and these tones can also be dynamically allocated. The advantage of OFDM is the flexibility in assignment and the ability to dynamically change it as well as data rates on the tones depending on the channel condition of the different users.

9.2.3 Code Division Multiple Access (CDMA)

CDMA is philosophically slightly different from TDMA and FDMA in that the users do interfere with one another since they are allowed to transmit simultaneously. The idea is to assign a "spreading code" to each user so as to minimize the interference. There are two flavors of CDMA - frequency hopped CDMA and direct-sequence CDMA. Both arise from "spread-spectrum" techniques which transmit over a wide bandwidth and only transmit at a rate much smaller than the transmission bandwidth could have allowed.

- **Frequency-hopped spread spectrum:** Here we can imagine a combination of FDMA and TDMA, where a random assignment of frequency and time slots is given to each user. This is illustrated in Figure 9.5 in a time frequency grid.

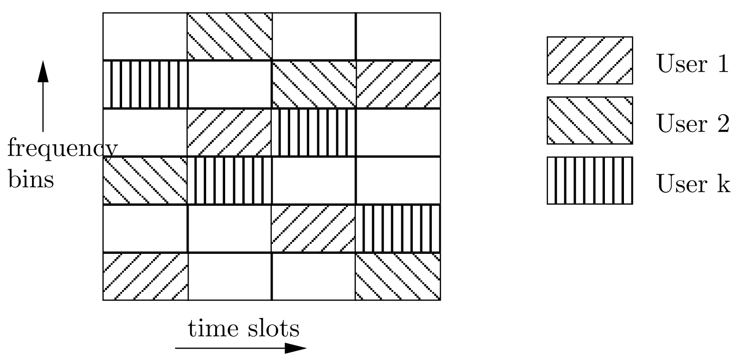


Figure 9.5: Frequency hopped spread Spectrum.

The users' "hopping pattern" are basically the frequency bin they transmit in at a certain time. Therefore, if they are assigned random hopping patterns, they could in principle "collide" *i.e.* get assigned same transmission slot. Also note that they transmit over much larger bandwidth overall, than their data rate requires, *i.e.*, they use many more degrees of freedom than their transmission needs. This gives it the name spread-spectrum, since the signals spread over a large bandwidth. This spreading over a wide bandwidth is in order to accommodate other simultaneously transmitting users. This form of frequency-hopping is also used in some modern TDMA/FDMA systems in order to get frequency diversity as well as reduce interference to other geographical cells using the same frequency spectrum. For example GSM (the European cellular standard) uses such a technique.

- **Direct-sequence spread spectrum:** In this system, unlike in frequency-hopped spread spectrum, the users intentionally interfere with one another. The basic structure of a direct sequence spread spectrum system is shown in Figure 9.6 . The information to be transmitted is modulated by a pseudo-noise sequence and transmitted over a wide bandwidth.

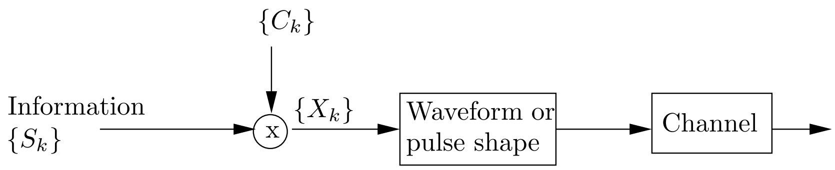


Figure 9.6: Direct sequence spread Spectrum.

Typically, the symbol or information sequence is at a much lower rate than the variation in the modulating pseudo-noise sequence. Therefore we obtain

$$x_k = c_k s \quad k = 0, \dots, L-1.$$

i.e., a frame of L symbols is generated for every information symbol time creating a spreading of the information symbol over L times its original bandwidth.

Therefore for a particular block one gets the transmitted symbols (see Figure 9.7)

$$\begin{bmatrix} x_0^{(b)} \\ \vdots \\ x_{L-1}^{(b)} \end{bmatrix} = \begin{bmatrix} c_0^{(b)} \\ \vdots \\ c_{L-1}^{(b)} \end{bmatrix} s^{(b)} \tag{9.1}$$

This is transmitted at the *same* time-frame as the original signal, *i.e.,* the bandwidth is L times the bandwidth of the information sequence $\{S^{(b)}\}$.

The "spreading code" of the pseudo-noise sequence $\{c_k^{(b)}\}$ could be the same for every block, *i.e.* $c_k^{(b)} = c_k^{(b+1)}, \;\; k = 0, \dots, L-1$ and this is called "short codes". Or they could vary from block to block and this called "long codes". The long codes are used in the Qualcomm IS-95 CDMA system which is used in several commercial mobile telephony systems. The transmission of the spread symbols over a shared channel is based on the supposition that the different users get *distinct* random spreading codes and therefore look like noise to one another. Another jargon used is that the pseudo-noise sequence occurs at a period of T_c known as "chip period" and this is chosen to be T/L where T is the period of the original symbol sequence. Therefore the new symbol period is T/L causing the bandwidth to become L times the

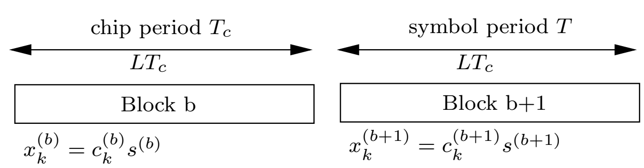


Figure 9.7: Frame format of DS-CDMA.

original bandwidth. The power per symbol transmitted is reduced by a factor of L and therefore reduce the interference to the other users. In a multiple access area broadcast configuration, by considering the other users as undesired noise-like interference, one can see that a simple single user receiver can be built. We will study more sophisticated receivers that take into account presence of multiple users. Note that, "spreading gain" refers to the length L of the spreading code.

9.3 Direct-sequence CDMA multiple access channels

We will focus on one hierarchical configuration - the many-to-one communication of the wireless "uplink" (see Figure 9.2). In the TDMA and FDMA access techniques, the users do *not* interfere with one another and therefore the detection techniques discussed in the first part of the class (for single-user communication) apply directly and there is no need for further discussion. The problem becomes more interesting for direct sequence CDMA (or spread spectrum) where the users continuously interfere with one another.

9.3.1 DS-CDMA model

Recall from (9.1) that in Direct-sequence CDMA (DS-CDMA) each user u is assigned a distinct spreading code for each information block b, *i.e.,* a code sequence

$$\left\{c_u^{(b)}(0), \ldots, c_u^{(b)}(L-1)\right\},$$

for user u, in block b, of length L. Therefore each user modulates its information sequence $s_u^{(b)}$ through its spreading sequence to produce

$$\begin{aligned} x_u^{(b)}(l) &= c_u^{(b)}(l) s_u^{(b)}, l = 0, \ldots, L-1 \\ \mathbf{x}_u^{(b)} &= \begin{bmatrix} x_u^{(b)}(0) \\ \vdots \\ x_u^{(b)}(L-1) \end{bmatrix} = \begin{bmatrix} c_u^{(b)}(0) \\ \vdots \\ c_u^{(b)}(L-1) \end{bmatrix} s_u^{(b)} \end{aligned} \tag{9.2}$$

Given the multiple access configuration (see figure 9.2), each user u transmits to the (central) base-station.

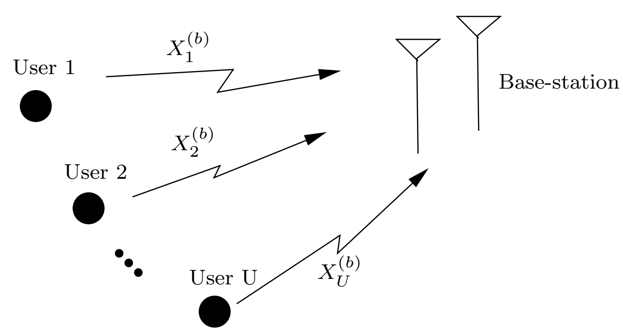


Figure 9.8: DS-CDMA multiple access channel

Each user experiences its own fading channel $h_u^{(b)}(l)$, $l = 0, \ldots, L-1$. For simplicity we make the following assumptions.

Assumption # 1: All users are synchronized both at block and at chip level, *i.e.* their transmission frames coincide exactly.
Assumption # 2: The power of each user is adjusted such that the channel fading (or attenuation) is exactly compensated.

This assumption implies that we can consider a unit gain channel for each user. This can be accomplished by user "power control" where the users change their transmit power according to feedback given by the base-station. Such a mechanism is commonly used in commercial DS-CDMA systems to reduce the received power disparity among the users.
Assumption # 3: The delay spread in the channel is small in comparison to the symbol period T and therefore the channel does not give rise to intersymbol interference. This assumption is mainly for simplicity and can be easily taken care of with more sophisticated receivers. In order to illustrate the concepts we will not deal with this issue right now.

Under these assumptions we can write the received signal at the base-station as (see figure 9.2 and 9.8)

$$\begin{aligned} y^{(b)}(t) &= \sum_{u=1}^{U} s_u^{(b)} \varphi_u(t) + z^{(b)}(t),\ t \in ((b-1)T, bT) \\ \varphi_u^{(b)}(t) &= \sum_{l=0}^{L-1} c_u^{(b)}(l)\psi(t - lT_c),\ t \in ((b-1)T, bT) \end{aligned} \tag{9.3}$$

where $\mathbf{z}^{(b)}(t)$ is additive (Gaussian) receiver noise.

The basis expansion $\varphi_u(t)$ for each user is in terms of a waveform at the "chip" level, *i.e,* $\psi(t)$ shifted at "chip" period T_c. The symbol period is $T = LT_c$, where L is the spreading gain. The waveform $\varphi_u(t)$ serves as a *signature waveform* for user u.

9.3.2 Multiuser matched filter

The continuous time waveform equation (9.3) is the same type of relationship as we encountered in the single-user case. Therefore, the optimal detector for this can be found by considering this as a larger dimensional single-user problem, *i.e.* $\mathbf{s}^{(b)} = [s_1^{(b)}, \ldots, s_U^{(b)}]$ as the input symbol expressed in the basis function $\varphi^{(b)}(t) = [\varphi_1^{(b)}(t), \ldots, \varphi_U^{(b)}(t)]$. Then as shown in Section 4.2.1 we have seen that the optimum receiver is a bank of matched filters as shown in Figure 9.9.

Therefore we can write the output of the matched as,

$$\begin{aligned} y_1^{(b)} &= \int_{(b-1)T}^{bT} y^{(b)}(t)\varphi_1^{(b)*}(t)dt = \sum_{u=1}^{U} s_u^{(b)} < \varphi_u^{(b)}, \varphi_1^{(b)} > + \underbrace{< z^{(b)}, \varphi_1^{(b)} >}_{Z_1^{(b)}} \\ &\vdots \\ y_U^{(b)} &= \int_{(b-1)T}^{bT} y^{(b)}(t)\varphi_U^{(b)*}(t)dt = \sum_{u=1}^{U} s_u^{(b)} < \varphi_u^{(b)}, \varphi_U^{(b)} > + \underbrace{< z^{(b)}, \varphi_U^{(b)} >}_{z_U^{(b)}} \end{aligned}$$

Hence, we see that

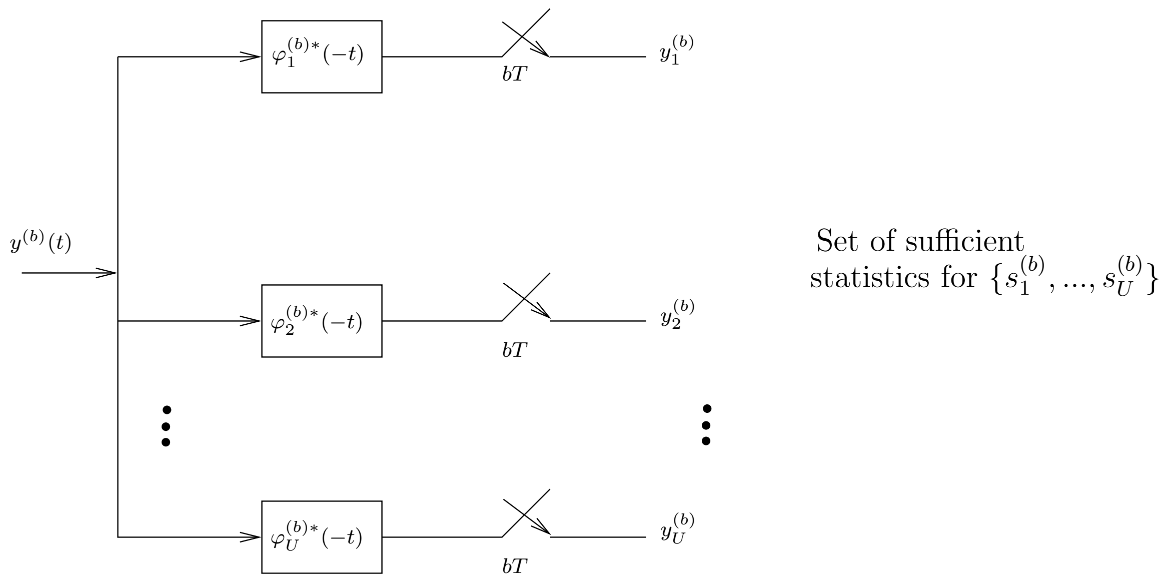


Figure 9.9: Multiuser matched filter.

$$
\begin{aligned}
y_1^{(b)} &= \sum_{u=1}^{U} s_u^{(b)} \rho_{u,1}^{(b)} + z_1^{(b)} \\
&\vdots \\
y_U^{(b)} &= \sum_{u=1}^{U} s_u^{(b)} \rho_{u,U}^{(b)} + z_U^{(b)}
\end{aligned}
$$

where

$$\rho_{u,l} =< \varphi_u^{(b)}, \varphi_l^{(b)} >= \int_{(b-1)T}^{bT} \varphi_u^{(b)}(t) \varphi_l^{(b)*}(t) dt$$

Note that as proved in Section 4.2.1, these form a set of sufficient statistics to detect the symbols $\{\mathbf{s}^{(b)}\}$ where $\mathbf{s}^{(b)} = [s_1^{(b)}, ..., s_V^{(b)}]$.
Hence for the received signal block $t \in [(b-1)T, bT]$ we get

$$\mathbf{y}^{(b)} = \begin{bmatrix} y_1^{(b)} \\ \vdots \\ y_u^{(b)} \end{bmatrix} = \underbrace{\begin{bmatrix} \rho_{1,1}^{(b)} & \dots & \rho_{U,1}^{(b)} \\ \vdots & & \vdots \\ \rho_{1,U}^{(b)} & \dots & \rho_{U,U}^{(b)} \end{bmatrix}}_{\mathbf{R}^{(b)}} \underbrace{\begin{bmatrix} s_1^{(b)} \\ \vdots \\ s_U^{(b)} \end{bmatrix}}_{\mathbf{s}^{(b)}} + \underbrace{\begin{bmatrix} Z_1^{(b)} \\ \vdots \\ Z_U^{(b)} \end{bmatrix}}_{\mathbf{z}^{(b)}}$$

This can be compactly written as,

$$\mathbf{y}^{(b)} = \mathbf{R}^{(b)} \mathbf{s}^{(b)} + \mathbf{z}^{(b)} \tag{9.4}$$

This relationship is the starting point for multiuser detection. Note that $\mathbb{E}\left[\mathbf{z}^{(b)} \mathbf{z}^{(b)*}\right] = \mathbf{R}^{(b)} \sigma^2$, if $\mathbf{z}^{(b)}(t)$ is AWGN with variance σ^2.
Notes:

1. The form and structure of the receiver is very similar to the single user case. The difference is that there are U information sequences to be recovered (one for every user). The conversion of the waveform to discrete-time through collecting sufficient statistics is identical to the simple-user case and we do not dwell on this issue further.

   This is an important point that should not be neglected. As in the single-user presentation we start from the discrete-time version here as well.

2. In the relationship (9.4), we can add more details such as the channel attenuations $\{h_u^{(b)}(l)\}$, delayed symbols causing inter-symbol interference as well as asynchronism between users. However, conceptually all these are handled in a similar manner and therefore for simplicity of exposition, these are ignored.

3. If short codes are employed, $\mathbf{c}^{(b)}$ would be independent of the block index b. However for long codes, this dependence exists and therefore the model in (9.4) applies to both cases.

4. The linear relationship in (9.4) is reminiscent of the single-user ISI channel relationship. Therefore one would expect schemes to detect $\mathbf{s}^{(b)}$ would also be similar. In fact all multiuser detections have an analogous ISI detector structure. Here we consider $[s_1^{(b)}, \dots, s_U^{(b)}]$ as a "sequence" of vector information symbols.

5. In the absence of ISI or asynchronism, we can do symbol-by-symbol detection, *i.e.,* we detect $\mathbf{s}^{(b)} = [s_1^{(b)}, \dots, s_U^{(b)}]$ by considering only $\mathbf{y}^{(b)}$ and therefore for brevity of notation we will drop the index b.

6. The optimal detector would have complexity $O(|\mathcal{S}|^U)$ where $\mathcal{S}$ is the size of the constellation (common for each user for simplicity). All the principles of optimal detection we developed for the single user case apply here by considering the symbol to be detected as $\begin{bmatrix} s_1^{(b)} \\ \vdots \\ s_U^{(b)} \end{bmatrix}$ and therefore we will not repeat it here (*e.g.* MAP, ML, etc.).

7. Where there is a large number of users U, the complexity grows exponentially in U and therefore we examine sub-optimal detectors. The reason is almost identical to the cause for looking at suboptimal equalizers of lower complexity for ISI channels. In fact the mathematical model given in (9.4) is identical in form to that in the ISI model we had developed for the finite block case.

9.4 Linear Multiuser Detection

Motivated by the last comment in Section 9.2.2, we examine sub-optimal linear detectors for the DS-CDMA multiple access channel. We assume that the symbols are i.i.d. and therefore we can do symbol-by-symbol detection rather than sequences detection.
Let us rewrite (9.4) here without the cumbersome additional block index b. This is because we know that "symbol-by-symbol" detection is optimal for this case and we develop linear "symbol-by-symbol" detections.

$$\mathbf{y} = \mathbf{R}\mathbf{s} + \mathbf{z} \tag{9.5}$$

The model is similar to the finite block ISI model and we considered two types of linear detections of lower complexity, *i.e.*, the zero-forcing and the MMSE receivers. In the multiuser case as well, the analogous detectors are called the "decorrelating receivers" and the "MMSE linear detector".

9.4.1 Decorrelating receiver

In analogy with the ISI case, we want to recover a particular symbol s_u from the interfering symbols in (9.5). Therefore in the zero-forcing case we invert the "channel". In this case the role of the channel is taken by the cross-correlation matrix $\mathbf{R}$.

$$\mathbf{R}^{-1}\mathbf{y} = \mathbf{s} + \mathbf{R}^{-1}\mathbf{z}$$

and then we can find the symbol for each user u as,

$$\hat{\mathbf{s}}_u = \mathcal{D}[(\mathbf{R}^{-1}\mathbf{y})_u]$$

where $\mathcal{D}$ is a single-user detector for the given Gaussian noise (see also Figure 9.10).

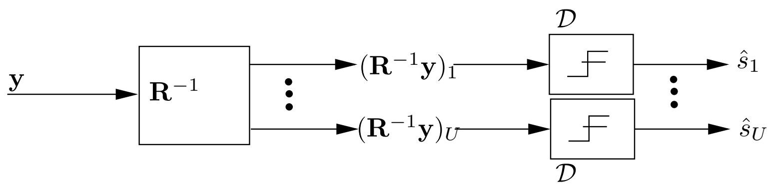


Figure 9.10: Decorrelating multiuser detector

This receiver has the same attributes as the zero forcing detector equalizer, *i.e.,* it is conceptually simple and ignores the background noise. Therefore it could cause noise enhancement especially if $\mathbf{R}$ is close to being singular. Therefore, we can take the next step of accounting for the presence of noise and develop the MMSE linear multiuser detector.

9.4.2 MMSE linear multiuser detector

We want to find a linear transformation $\mathbf{M}$ such that output is "close" to the information symbols, *i.e.*, $\mathbf{My}$ is close to $\mathbf{s}$.
The meaning of "close" is make precise by considering the MMSE metric and therefore we pose the problem as

$$\min_{\mathbf{M}\in\mathbb{C}^{U\times U}} \mathbb{E}\left[||\mathbf{s}-\mathbf{My}||^2\right]$$

Now, by using orthogonality principle, we get

$$\mathbb{E}\left[(\mathbf{s}-\mathbf{My})\mathbf{y}^*\right] = 0,$$

This yields

$$\begin{aligned}
\mathbb{E}\left[\mathbf{sy}^*\right] &= \mathbf{M}\mathbb{E}\left[\mathbf{yy}^*\right] \\
\mathbf{M} &= \mathbb{E}\left[\mathbf{sy}^*\right]\left[\mathbb{E}\left[\mathbf{yy}^*\right]\right]^{-1} \qquad (9.6)
\end{aligned}$$

Now

$$\begin{aligned}
\mathbb{E}\left[\mathbf{sy}^*\right] &= \mathbb{E}\left[\mathbf{ss}^*\right]\mathbf{R}^* = E_s\mathbf{I}\mathbf{R}^* \\
\mathbb{E}\left[\mathbf{yy}^*\right] &= \mathbf{R}\mathbb{E}\left[\mathbf{ss}^*\right]\mathbf{R}^* + \mathbb{E}\left[\mathbf{zz}^*\right] \\
&= E_s\mathbf{RR}^* + \sigma^2\mathbf{R} \qquad (9.7)
\end{aligned}$$

where we have assumed equal power E_s for each user and independent symbols, *i.e.* $\mathbb{E}[\mathbf{s}\mathbf{s}^*] = E_s\mathbf{I}$. Moreover since the noise $\mathbf{z}$ is the output of the matched filter (see Figure 9.11) and equation (9.4), the noise has covariance $\sigma^2\mathbf{R} = \mathbb{E}[\mathbf{z}\mathbf{z}^*]$.
Using (9.7) in (9.6) we get,

$$\mathbf{M} = E_s\mathbf{R}^*(E_s\mathbf{R}\mathbf{R}^* + \sigma^2\mathbf{R})^{-1}$$

Since $\mathbf{R}$ is a cross-correlation matrix of the signature waveforms, (9.4), it is symmetric, *i.e.,* $\mathbf{R}^* = \mathbf{R}$ and therefore

$$\mathbf{M} = \left(\mathbf{R} + \frac{\sigma^2}{E_s}\mathbf{I}\right)^{-1}$$

in the MMSE-optimal linear multiuser detector. The MMSE-linear multiuser detector is depicted in Figure 9.11.

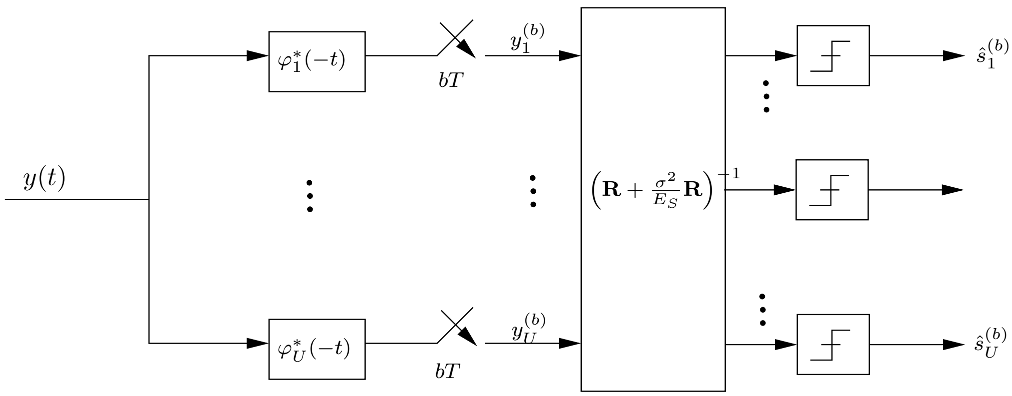


Figure 9.11: Receiver for linear multiuser detection.

Therefore even if $\mathbf{R}$ is singular, $\mathbf{M}$ is well defined and does not cause noise enhancement. Moreover just as in the linear equalizers if we let $\sigma^2/E_s \to 0$ the MMSE linear detector reduces to the decorrelating receiver.

Notes:

1. The main utility of multiuser detection is the resistance to the “near-far” problem. Here the users could have very disparate received power and the strong user could ”drown” out the weak user. All the forms of multiuser detections do exhibit resistance to this problem. We have not explicitly exhibited them since we assumed equal received power for all users. However a simple modification to (9.4) would take care of this. If each user u has received power $A_u^{(b)}$, then

$$\mathbf{y}^{(b)} = \underbrace{\begin{bmatrix} \rho_{1,1}^{(b)} & \cdots & \rho_{U,1}^{(b)} \\ \vdots & & \vdots \\ \rho_{1,U}^{(b)} & \cdots & \rho_{U,U}^{(b)} \end{bmatrix}}_{\mathbf{R}^{(b)}} \underbrace{\begin{bmatrix} A_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_U \end{bmatrix}}_{\mathbf{A}^{(b)}} \underbrace{\begin{bmatrix} s_1^{(b)} \\ \vdots \\ s_U^{(b)} \end{bmatrix}}_{\mathbf{s}^{(b)}} + \underbrace{\begin{bmatrix} z_1^{(b)} \\ \vdots \\ z_U^{(b)} \end{bmatrix}}_{\mathbf{z}^{(b)}}.$$

In more compact form,

$$\mathbf{y}^{(b)} = \mathbf{R}^{(b)}\mathbf{A}^{(b)}\mathbf{s}^{(b)} + \mathbf{z}^{(b)}$$

Then the (zero-forcing) decorrelating receiver remains

$$\mathbf{A}^{(b)}\hat{\mathbf{s}}^{(b)} = \mathbf{R}^{-1}\mathbf{y}^{(b)} = \mathbf{A}^{(b)}\mathbf{s}^{(b)} + \mathbf{R}^{-1}\mathbf{z}^{(b)}$$

and the MMSE linear multiuser detector becomes

$$\mathbf{M} = E_S \mathbf{A}\mathbf{R}\left(\mathbf{R}\mathbf{A}^2\mathbf{R} + \sigma^2\mathbf{R}\right)^{-1} = \mathbf{A}^{-1}\left(\mathbf{R} + \frac{\sigma^2}{E_S}\mathbf{A}^{-2}\right)^{-1}$$

where for simplicity $\mathbf{A}$ is assumed to be real. Therefore both receivers take into account the disparate received power and boost the weak user in order not to be drowned out by the stronger user.

2. One can also develop decision-driven multiuser detectors analogous to the DFE studied earlier. These are called successive cancellation receivers and have a rich history. Here the decisions of some users are used to eliminate them and help decoding the other users. A crucial issue here is the order in which the users are decoded. This topic is beyond the scope of this chapter.

3. Detector structures for asynchronous CDMA channels can be developed by realizing that asynchronism leads to inter-symbol interference between successive symbols of the different users. Almost the same principles can be applied to develop receivers for this case. Again the details are beyond the scope of this chapter.

9.5 Epilogue for multiuser wireless communications

We mainly studied one configuration (illustrated in figure 9.2) among several in wireless networks. The topic of multiuser wireless communication is still an active research topic and the more advanced topologies are beyond the scope of this class.
The broadcast (downlink) configuration (illustrated in figure 9.1) is of particular importance and we did not cover it in much detail in this class. Here the crucial difference from multiple access channels is that each user has access to only its received signal and therefore cannot do joint detection, *i.e.,* cooperate with other users. This crucial difference makes the broadcast channel much more challenging and there are several open questions on this topic.
Finally ad hoc networks have become increasingly important in the recent past and have been a source of several important new developments. This topic is again outside the scope of this chapter and is still a developing research story.

9.6 Problems

Problem 9.1

Let us examine "symbol-by-symbol" synchronus multiuser receiver performance over the following channel:

$$\mathbf{y} = \mathbf{R}\mathbf{A}\mathbf{s} + \mathbf{z}$$

where $\mathbf{y}$ is the received signal, $\mathbf{R}$ is the channel cross correlation matrix, $\mathbf{A}$ is the power assignment matrix, $\mathbf{s}$ are the transmitted symbols and $\mathbf{z}$ is Gaussian additive noise with covariance matrix $\sigma^2\mathbf{R}$.

1. Find the SINR for the decorrelating receiver. Identify the source of the noise-enhancement.

2. Find the SINR for the MMSE linear multiuser detector. Compare this to the MMSE linear equalizer. What happens when $\mathbf{R}$ is close to being singular?

Problem 9.2

Consider transmission using Direct-sequence CDMA (DS-CDMA) using block invariant code sequences for each user. Then the output during a block period is given by

$$y(t) = \sum_{u=1}^{U} s_u \phi_u(t) + z(t), \quad t \in ((b-1)T, bT),$$

$$\phi_u(t) = \sum_{l=0}^{L-1} c_u(l)\psi\left(t - lT_c\right), \quad t \in ((b-1)T, bT).$$

Assume that shifted versions of $\psi(t)$ by some integral multiple of T_c are orthogonal to each other, i.e.,

$$\langle \psi\left(t - mT_c\right), \psi\left(t - nT_c\right) \rangle = \begin{cases} 1 & m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the entries of correlation matrix $\mathbf{R}$ are given by the dot product of the code-sequences i.e.,

$$\mathbf{R}_{ij} = \sum_{l=0}^{L-1} c_i(l) c_j^*(l).$$

Problem 9.3

In this problem we study the *near-far* problem: users 1 and 2 are transmitting and we want to detect user 1 using conventional detection (single user detection). If the user 2 is sufficiently powerful then it can cause arbitrary performance degradation in the detection of user 1. To this end, consider a two-user, synchronous DS-CDMA system with signature sequences given by

$$\begin{aligned} s_1(t) &= \begin{cases} 1 & 0 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases} \\ s_2(t) &= \begin{cases} A & 0 \le t \le 0.5, \\ -AB & 0.5 < t \le 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The parameters take values $A \ge 0$ and $0 \le B \le 1$. The received waveform at time t can be written as

$$r(t) = \sum_{k=1}^{2} \sum_{i=-\infty}^{\infty} \sqrt{E_k}\, b_k(i) s_k(t-i) + n(t)$$

where $n(t)$ is the white Gaussian noise with power spectral density σ^2, and $b_k(i) = \pm 1$ with equal probability. A conventional detector is used, which detects the $b_k(0)$ bit as

$$\hat{b}_k(0) = \operatorname{sgn}(r_k), \quad \text{where} \quad r_k = \int_0^1 r(t) s_k(t) dt.$$

1. Determine expressions for the probability of bit error for user 1 and 2 respectively.
2. For user 1, what is the form of the bit error expression for $A \to 0, B < 1$.
3. For user 1, what is the form of the bit error expression for $A \to \infty, B < 1$. What does this say about the conventional detection?
4. For user 1, what is the form of the bit error expression for $B = 1$. Compare with the result in part(3) and explain.

Problem 9.4

Suppose we have two users $u = 1, 2$, transmitting information $\{x_1(k)\}$ and $\{x_2(k)\}$. Consider a simple multiple access channel where the received signal (discrete time) is

$$y(k) = x_1(k) + x_2(k) + z(k)$$

where $\{z(k)\}$ is additive white complex Gaussian noise with variance σ^2, is i.i.d. and independent of $\{x_1(k)\}$ and $\{x_2(k)\}$. You may assume that x_1 and x_2 are independent with identical variance $\mathcal{E}_x$.

1. Suppose we use blocks of length two for transmission and the users use the following transmission strategy,

$$\left.\begin{array}{rclcrcl} x_1(k) & = & s_1 & , & x_1(k+1) & = & s_1 \\ x_2(k) & = & -s_2 & , & x_2(k+1) & = & s_2 \end{array}\right\} \tag{9.8}$$

   Express the received signal, $[y(k),\, y(k+1)]$ in terms of the transmitted symbols, i.e. specialize

$$\begin{array}{rcl} y(k) & = & x_1(k) + x_2(k) + z(k) \\ y(k+1) & = & x_1(k+1) + x_2(k+1) + z(k+1) \end{array}$$

   to the transmission strategy in (9.8), prove that

$$\mathbf{y} = \mathbf{R}\mathbf{s} + \mathbf{z}.$$

   and find the form of $\mathbf{R}$.

2. Find

$$\tilde{\mathbf{y}} = \mathbf{R}^*\mathbf{y}$$

   and comment about its implications to detecting s_1 and s_2. Is this equivalent to the decorrelating detector?

3. Find the MMSE multiuser detector $\mathbf{M}$ such that

$$\mathbb{E}||\mathbf{M}\mathbf{y} - \mathbf{s}||^2 \quad \text{is minimized.}$$

   Explicitly calculate $\mathbf{M}$ and comment about its relationship to the receiver in (2).

4. What is the error performance of the receivers in (2) and (3) in the presence of i.i.d. noise $\mathbb{C}\eta(0, \sigma^2)$?

5. What is the transmission rate for each of the users in the model given in (9.8)?

6. More generally if we develop a transmission strategy for U users, such that

$$\mathbf{y} = \mathbf{R}\mathbf{s} + \mathbf{z}$$

   where $\mathbf{y}, \mathbf{z} \in \mathbb{C}^{U \times U}$, $\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_U \end{bmatrix} \in \mathbb{C}^U$ and $\mathbf{R}^*\mathbf{R} = U\mathbf{I}$. Develop the decorrelating and the MMSE linear multiuser detector for this case. You may assume $\mathbb{E}[\mathbf{s}\mathbf{s}^*] = \mathcal{E}_s\mathbf{I}$, $\mathbb{E}[\mathbf{z}\mathbf{z}^*] = \sigma^2\mathbf{I}$ and that $\mathbf{s}$ and $\mathbf{z}$ are independent. Can you comment about the relationship of this strategy with respect to TDMA or FDMA?

Problem 9.5

Consider a three-user synchronous CDMA channel with equal power users $A_1 = A_2 = A_3 = 1$.

1. Find a crosscorrelation matrix $\mathbf{R}$ such that:

$$\frac{|([\mathbf{R}+\sigma^2\mathbf{I}]^{-1}\mathbf{R})_{13}|}{([\mathbf{R}+\sigma^2\mathbf{I}]^{-1}\mathbf{R})_{11}} > |\rho_{13}|$$

   Without loss of generality, one can consider that det$\mathbf{R} = 0$, $\rho_{13} = 0$, $\rho_{12} = \rho_{23}$ and $\rho_{11} = 1$. What can one conclude?

2. For the crosscorrelation matrix you selected above, verify that:

$$\frac{([\mathbf{R}+\sigma^2\mathbf{I}]^{-1}\mathbf{R})_{12}^2}{([\mathbf{R}+\sigma^2\mathbf{I}]^{-1}\mathbf{R})_{11}^2} + \frac{([\mathbf{R}+\sigma^2\mathbf{I}]^{-1}\mathbf{R})_{13}^2}{([\mathbf{R}+\sigma^2\mathbf{I}]^{-1}\mathbf{R})_{11}^2} < \rho_{12}^2 + \rho_{13}^2$$

   What can one conclude?

Problem 9.6

[Multiple Access Channel and Transmit Code] Consider a two user multiple access channel where each user has two transmit antennas and the receiver has two antennas (see Figure 9.6). Let

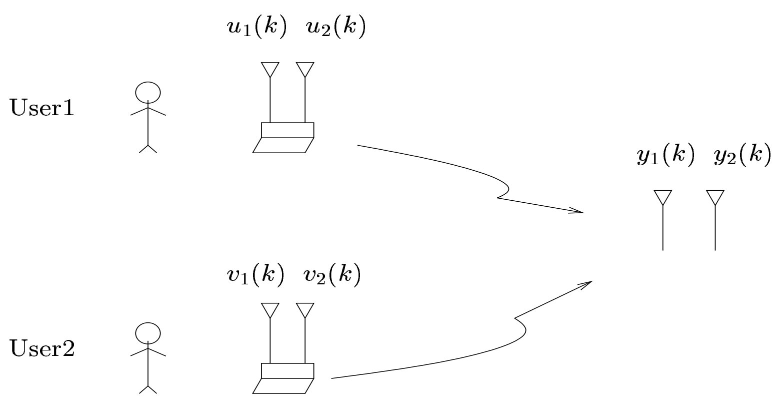


Figure 9.12: Multiple access channel for problem 4.

$\mathbf{u}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$, $\mathbf{v}(k) = \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$ be the transmit signal from each user. The receiver gets a linear combination of the transmit signals,

$$y_1(k) = e_1 u_1(k) + e_2 u_2(k) + f_1 v_1(k) + f_2 v_2(k) + z_1(k),$$

$$y_2(k) = h_1 u_1(k) + h_2 u_2(k) + g_1 v_1(k) + g_2 v_2(k) + z_2(k),$$

where $z_i(k) \sim \mathbb{C}\eta(0, \sigma_z^2)$ and $\{z_i(k)\}$ is i.i.d circularly symmetric Gaussian random variables. Now, suppose the users use an Alamouti code, *i.e.,* for user 1 the transmit signal is

$$u_1(k) = a_1, \quad u_2(k) = a_2,$$

$$u_1(k+1) = -a_2^*, \quad u_2(k+1) = a_1^*,$$

and for user 2, it is

$$v_1(k) = b_1, \quad v_2(k) = b_2,$$
$$v_1(k+1) = -b_2^*, \quad v_2(k+1) = b_1^*.$$

Therefore we get

$$\begin{bmatrix} y_1(k) & y_1(k+1) \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{bmatrix} + \begin{bmatrix} z_1(k) & z_1(k+1) \end{bmatrix} \tag{9.9}$$

$$\begin{bmatrix} y_2(k) & y_2(k+1) \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{bmatrix} + \begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{bmatrix} + \begin{bmatrix} z_2(k) & z_2(k+1) \end{bmatrix}. \tag{9.10}$$

(a) Prove that (9.9) can be equivalently rewritten as

$$\tilde{\mathbf{Y}}_\mathbf{1} = \begin{bmatrix} \tilde{y_1}(k) & \tilde{y_1}(k+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} e_1 & -e_2^* \\ e_2 & e_1^* \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} f_1 & -f_2^* \\ f_2 & f_1^* \end{bmatrix} + \begin{bmatrix} \tilde{z}_1(k) & \tilde{z}_1(k+1) \end{bmatrix}$$

and (9.10) can be rewritten as

$$\tilde{\mathbf{Y}}_\mathbf{2} = \begin{bmatrix} \tilde{y_2}(k) & \tilde{y_2}(k+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} g_1 & -g_2^* \\ g_2 & g_1^* \end{bmatrix} + \begin{bmatrix} \tilde{z}_2(k) & \tilde{z}_2(k+1) \end{bmatrix},$$

where $\tilde{z}_1(k), \tilde{z}_1(k+1), \tilde{z}_2(k), \tilde{z}_2(k+1)$ are i.i.d circularly symmetric complex Gaussian random variables with distribution $\mathbb{C}\eta(0, \sigma_z^2)$. Explicitly write out how $\begin{bmatrix} \tilde{y_1}(k) & \tilde{y_1}(k+1) \end{bmatrix}$, $\begin{bmatrix} \tilde{y_2}(k) & \tilde{y_2}(k+1) \end{bmatrix}$ are related to $\begin{bmatrix} y_1(k) & y_1(k+1) \end{bmatrix}$, $\begin{bmatrix} y_2(k) & y_2(k+1) \end{bmatrix}$ respectively.

(b) Let

$$\mathbf{E} = \begin{bmatrix} e_1 & -e_2^* \\ e_2 & e_1^* \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 & -f_2^* \\ f_2 & f_1^* \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g_1 & -g_2^* \\ g_2 & g_1^* \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} h_1 & -h_2^* \\ h_2 & h_1^* \end{bmatrix},$$

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}, \quad \tilde{\mathbf{Z}}_\mathbf{1} = \begin{bmatrix} \tilde{z}_1(k) & \tilde{z}_1(k+1) \end{bmatrix}, \quad \tilde{\mathbf{Z}}_\mathbf{2} = \begin{bmatrix} \tilde{z}_2(k) & \tilde{z}_2(k+1) \end{bmatrix}.$$

Then the equations (9.9,9.10) can be written as

$$\begin{bmatrix} \tilde{\mathbf{Y}}_\mathbf{1} & \tilde{\mathbf{Y}}_\mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{H} \\ \mathbf{F} & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{Z}}_\mathbf{1} & \tilde{\mathbf{Z}}_\mathbf{2} \end{bmatrix}.$$

Prove that

$$\mathbf{W} = \begin{bmatrix} \mathbf{I_2} & -\mathbf{E^{-1}H} \\ -\mathbf{G^{-1}F} & \mathbf{I_2} \end{bmatrix}$$

decouples the signals from user 1 and 2, *i.e.,*

$$\begin{bmatrix} \check{\mathbf{Y}}_\mathbf{1} & \check{\mathbf{Y}}_\mathbf{2} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{Y}}_\mathbf{1} & \tilde{\mathbf{Y}}_\mathbf{2} \end{bmatrix} \mathbf{W} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{bmatrix} + \begin{bmatrix} \check{\mathbf{Z}}_\mathbf{1} & \check{\mathbf{Z}}_\mathbf{2} \end{bmatrix}, \tag{9.11}$$

where $\check{\mathbf{Z}}_\mathbf{1}, \check{\mathbf{Z}}_\mathbf{2}$ are still Gaussian.

(c) Prove that $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$ are of the form

$$\tilde{\mathbf{H}} = \begin{bmatrix} \tilde{h}_1 & -\tilde{h}_2^* \\ \tilde{h}_2 & \tilde{h}_1^* \end{bmatrix}, \tilde{\mathbf{G}} = \begin{bmatrix} \tilde{g}_1 & -\tilde{g}_2^* \\ \tilde{g}_2 & \tilde{g}_1^* \end{bmatrix}.$$

Hint: You do not need to explicitly write out the expressions for $\tilde{h}_1, \tilde{h}_2, \tilde{g}_1, \tilde{g}_2$.

(d) In (9.11) it is seen that

$$\check{\mathbf{Y}}_{\mathbf{1}} = \mathbf{a}\tilde{\mathbf{H}} + \check{\mathbf{Z}}_{\mathbf{1}},$$
$$\check{\mathbf{Y}}_{\mathbf{2}} = \mathbf{b}\tilde{\mathbf{G}} + \check{\mathbf{Z}}_{\mathbf{2}}.$$

Show that

$$\check{\mathbf{Y}}_{\mathbf{1}}\tilde{\mathbf{H}}^* = ||\tilde{\mathbf{h}}||^2 \begin{bmatrix} a_1 & a_2 \end{bmatrix} + \check{\mathbf{Z}}_{\mathbf{1}}\tilde{\mathbf{H}}^*,$$
$$\check{\mathbf{Y}}_{\mathbf{2}}\tilde{\mathbf{G}}^* = ||\tilde{\mathbf{g}}||^2 \begin{bmatrix} b_1 & b_2 \end{bmatrix} + \check{\mathbf{Z}}_{\mathbf{2}}\tilde{\mathbf{G}}^*,$$

where $\tilde{\mathbf{h}} = \begin{bmatrix} \tilde{h}_1 & \tilde{h}_2 \end{bmatrix}$, $\tilde{\mathbf{g}} = \begin{bmatrix} \tilde{g}_1 & \tilde{g}_2 \end{bmatrix}$. This completes the decoupling of the individual streams of the multiple access channel.

(e) If $h_1, h_2, g_1, g_2, e_1, e_2, f_1, f_2$ are i.i.d and have distribution $\mathbb{C}\eta(0,1)$, can you guess the diversity order for detecting a_1, a_2, b_1, b_2?

Problem 9.7

[Relay Diversity] In a wireless network let us assume that there are three nodes, where the source

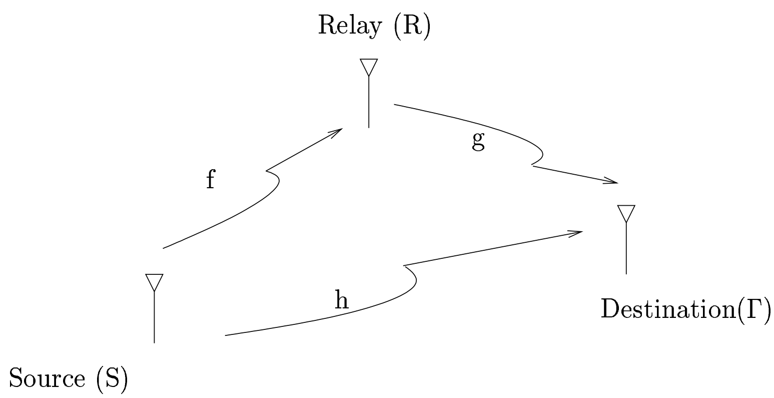


Figure 9.13: Communication using relay.

($\mathcal{S}$) wants to transmit information to the destination ($\mathcal{T}$) and can obtain help from a relay ($\mathcal{R}$). Assume that the channels are block time-invariant over a transmission block of size T. We use the following transmission protocol over a block of time of length T. Let $\{s(k)\} \in \{a, -a\}$ be a binary information

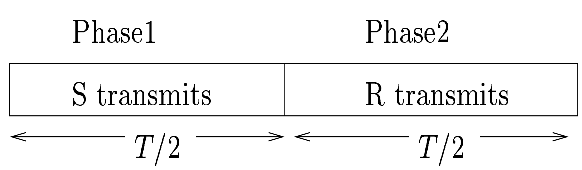


Figure 9.14: Transmission protocol.

sequence that source $\mathcal{S}$ wants to convey to the destination $\mathcal{T}$. Then, for the first phase the relay receives $y_{\mathcal{R}}(k)$ and the destination receives $y_{\mathcal{T}}^{(1)}(k)$, with

$$\begin{aligned} y_{\mathcal{R}}(k) &= fs(k) + z_{\mathcal{R}}(k), & (9.12) \\ y_{\mathcal{T}}^{(1)}(k) &= hs(k) + z_{\mathcal{T}}(k), & (9.13) \end{aligned}$$

where $z_{\mathcal{R}}(k), z_{\mathcal{T}}(k) \sim \mathbf{C}\eta(0, \sigma^2)$ are i.i.d, circularly symmetric complex Gaussian noise. We also assume a fading channel, $f, g, h \sim \mathbf{C}\eta(0,1)$ and are independent of each other. Assume that g, h are known at $\mathcal{T}$, f is known at $\mathcal{R}$, but they are unknown at $\mathcal{S}$ (except for part (c))
In the second phase for parts $(a), (b)$,

$$y_{\mathcal{T}}^{(2)}(k) = gu(k) + z_{\mathcal{T}}(k)$$

where $u(k)$ is the signal transmitted by the relay and $g \sim \mathbf{C}\eta(0,1)$.

(a) Suppose the relay was absent, *i.e.,* $u(k) = 0$, then give an expression (or bound) on the average error probability of $\{s(k)\}$ averaged over the channel realizations h. What is the diversity order, *i.e.,*

$$\lim_{\text{SNR}\to\infty} \frac{-\log(\bar{P}_e(\text{SNR}))}{\log(\text{SNR})}$$

for the detection of $\{s(k)\}$.

(b) Suppose that the relay $\mathcal{R}$ attempts to decode $\{s(k)\}$ in phase 1 and transmits $u(k) = \hat{s}_{\mathcal{R}}(k)$ in phase 2. That is, it sends the decoded sequence to $\mathcal{T}$. Assume now that there is an oracle which tells the destination $\mathcal{T}$ if relay $\mathcal{R}$ has decoded correctly or not. Note that the oracle just lets $\mathcal{T}$ know if $\hat{s}_{\mathcal{R}}(k) = s(k)$ but *not* its value. Now, $\mathcal{T}$ can use the received sequence from both phase 1 and phase 2 in order to decode $\{s(k)\}$. Find expressions (or bounds) for the error probability for this decoder averaged over the channel realizations which achieve the *best* diversity order at $\mathcal{T}$. What is the *best* diversity order that can be achieved at $\mathcal{T}$ for detecting $\{s(k)\}$? *Hint:* Develop a receiver strategy that uses the information given by the oracle. You do not need very detailed calculations for obtaining error probability bounds.

(c) Suppose now that we have a new protocol. Phase 1 is as before, where $\mathcal{S}$ transmits and both $\mathcal{R}$ and $\mathcal{T}$ receive the signal as in (9.12) and (9.13) respectively. At the end of phase 1, there is a feedback channel from $\mathcal{R}$ to $\mathcal{S}$ which informs $\mathcal{S}$ about the realization of channel f. Now the protocol $\mathcal{S}$ and $\mathcal{R}$ follow is given by: if $|f|^2 \leq c(\text{SNR})$ (where $c(\text{SNR})$ is a function of SNR), then in phase 2, $\mathcal{S}$ repeats the same information it transmitted in phase 1 and the relay $\mathcal{R}$ remains silent, *i.e.,* $\{s(k)\}$ from phase 1 is repeated and $u(k) = 0$ in phase 2. If $|f|^2 > c(\text{SNR})$, then the protocol is as in part (b), *i.e.,* $\mathcal{S}$ remains silent and $\mathcal{R}$ sends the decoded information $u(k) = \hat{s}_{\mathcal{R}}(k)$. Let $c(\text{SNR}) = \frac{1}{\text{SNR}^{1-\epsilon}}$ for an arbitrarily small $\epsilon > 0$, Assume that an oracle informs the receiver $\mathcal{T}$ whether in phase 2, $\mathcal{S}$ or $\mathcal{R}$ is transmitting. If in phase 2, $\mathcal{S}$ is transmitting, the receiver $\mathcal{T}$ forms the decision variable,

$$\tilde{y}_{\mathcal{T}} = \begin{bmatrix} h^* & h^* \end{bmatrix} \begin{bmatrix} y_{\mathcal{T}}^{(1)} \\ y_{\mathcal{T}}^{(2)} \end{bmatrix}$$

where $y_{\mathcal{T}}^{(1)}, y_{\mathcal{T}}^{(2)}$ are the received signals in phase 1 and phase 2 respectively. On the other hand if in phase 2, the relay $\mathcal{R}$ is transmitting, receiver $\mathcal{T}$ forms the decision variable

$$\tilde{y}_{\mathcal{T}} = \begin{bmatrix} h^* & g^* \end{bmatrix} \begin{bmatrix} y_{\mathcal{T}}^{(1)} \\ y_{\mathcal{T}}^{(2)} \end{bmatrix}$$

where again $y_{\mathcal{T}}^{(1)}, y_{\mathcal{T}}^{(2)}$ are the received signals in phase 1 and phase 2 respectively. The decision rule in both situations is that $\mathcal{T}$ chooses $\hat{s}_{\mathcal{T}} = +a$ if $\Re(\tilde{y}_{\mathcal{T}}) \geq 0$ and $\hat{s}_{\mathcal{T}} = -a$ otherwise. Analyze the performance of this receive strategy *i.e.,* find the diversity order that can be achieved by $\mathcal{T}$ for $\{s(k)\}$. Note that we are looking for diversity order and so we do not necessarily need a detailed analysis to find the diversity order. *Hint:* Condition on appropriate error events at the relay and use error probability bounds. You can also use properties given in the hints in the first page.

Part IV

Connections to Information Theory

Chapter 10

Reliable transmission for ISI channels

A focus of this class has been transmission and reception over LTI channels. Even in the wireless transmission, the channels were considered to be block-time-invariant. Therefore, we examine the fundamental limits of transmission over such channels and compare the structures we studied to that which information theory predicts. In order to do that, we first study the capacity of ISI channels in Section 10.1. Then in Section 10.2 and 10.3 we study canonical properties of OFDM and the MMSE-DFE as information theoretic structures.

10.1 Capacity of ISI channels

The complete and rigorous derivation of the capacity of a (bandlimited) frequency selective (time-invariant) channel is beyond the scope and goal of this class. Here we give a heuristic development of the capacity expression with the intention to connect it to the transmission and receiver structures that we have studied in this class.

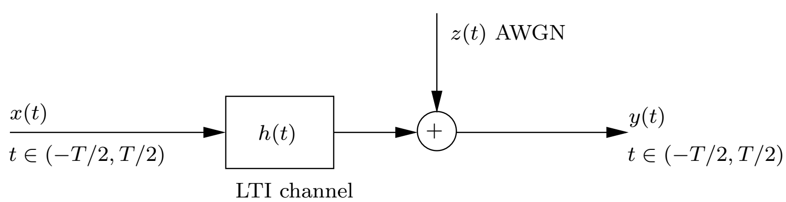


Figure 10.1: Model for frequency selective channels.

We represent the input $x(t)$ in terms of a Fourier series (where $x(t)$ is time-limited to $(-T/2, T/2)$ interval).

$$x(t) = \sum_{i=-\infty}^{+\infty} x_i \phi_i(t)$$

where $\{\phi_i\}$ are the Fourier series basis functions.

$$x(t) * h(t) = \int \sum_i x_i \phi_i(\tau) h(t-\tau) d\tau = \sum_i x_i \int \phi_i(\tau) h(t-\tau) d\tau$$

Let

$$\theta_i(t) = \int \phi_i(\tau) h(t-\tau) d\tau$$

Then

$$x(t) * h(t) = \sum_i x_i \theta_i(t)$$

$$y(t) = x(t) * h(t) + z(t) = \sum_i x_i \theta_i(t) + z(t)$$

When T becomes large, the Fourier series basis $\{\phi_i\}$ tend towards the Fourier basis, which are the eigenbasis for linear time-invariant channels. Therefore we can write

$$\theta_i(t) \approx H\left(\frac{2\pi}{T} i\right) g_T(t)$$

where $g_T(t)$ becomes the Dirac delta function when $T \to \infty$. Similarly, we can expand the noise in the frequency domain and by doing so we obtain:

$$y_i \approx x_i H\left(\frac{2\pi}{T} i\right) + z_i \tag{10.1}$$

where $\{y_i\}$ is the Fourier series coefficients of $y(t)$ and $\{z_i\}$ that of $z(t)$. This is a very imprecise and heuristic development and a precise development is clearly out of the scope of this class. Since the relationship in (10.1) is output of parallel additive Gaussian noise channels, mutual information is,

$$I(\{y_i\};\{x_i\}) = \frac{1}{2} \log \frac{|\mathbf{H}\mathbf{R}_x\mathbf{H}^t + \mathbf{R}_z|}{|\mathbf{R}_z|} \leq \frac{1}{2} \sum_i \log\left(\left|H\left(\frac{2\pi}{T} i\right)\right|^2 \frac{\mathbb{E}\left[|x_i|^2\right]}{\mathbb{E}\left[|z_i|^2\right]} + 1\right) \tag{10.2}$$

where $|.|$ denotes the determinant and we have assumed that $\mathbb{E}\left[z_i z_j\right] = \mathbb{E}\left[|z_i|^2\right] \delta_{i-j}$, *i.e.* white Gaussian noise. Now, if we impose the input power constraint,

$$\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \leq P$$

this translates (due to orthonormal expansion) to

$$\sum_i x_i^2 \leq PT$$

Now, the capacity therefore is maximization of the mutual information.

$$C_T = \max_{P_x : \sum x_i^2 \leq PT} \frac{1}{T} \sum_i \frac{1}{2} \log\left(\left|H\left(\frac{2\pi}{T} i\right)\right|^2 \frac{\mathbb{E}\left[|x_i|^2\right]}{\mathbb{E}\left[|z_i|^2\right]} + 1\right)$$

Since we can choose independent $\{x_i\}$, we can make the inequality in (10.2) into equality as we have done.

Now as $T \to \infty$, $\frac{1}{T}\mathbb{E}\left[|x_i|^2\right] = S_T\left(\frac{2\pi}{T} i\right) \to S(f)$ the power spectral density of the input, and $\frac{\mathbb{E}\left[|z_i^2|\right]}{T} \to \sigma^2$. Also $\frac{1}{T} \sum x_i^2 \to \int S(f) df$ and hence we get

$$\lim_{T\to\infty} C_T = \max_{S(f) : \int S(f) df \leq P} \frac{1}{2} \int \log\left(1 + |H(f)|^2 \frac{S(f)}{\sigma^2}\right) df$$

This is the expression for capacity that we will work with. Next the goal is to find the optimal input power spectrum.

$$C = \max_{S(f): \int S(f)df \le P} \frac{1}{2} \int \log\left(1 + \frac{|H(f)|^2}{\sigma^2} S(f)\right) df \tag{10.3}$$

Now, to find the maximizing input spectrum $S(f)$, we can do so by calculus of variations. However we can also do so by taking formal derivatives.

$$\max \frac{1}{2} \int \log\left(1 + \frac{|H(f)|^2}{\sigma^2} S(f)\right) df$$

such that

$$\int S(f)df \le P$$

Since C increases with $S(f)$, the last inequality can be replaced with equality. Let,

$$\mathcal{I} = \frac{1}{2} \int \log\left(1 + \frac{|H(f)|^2}{\sigma^2} S(f)\right) df - \lambda \int S(f)df$$

$$\frac{\partial \mathcal{I}}{\partial S(f)} = \frac{1/2}{1 + \frac{|H(f)|^2}{\sigma^2} S(f)} \frac{|H(f)|^2}{\sigma^2} - \lambda$$

The Kuhn-Tucker conditions imply that

$$\frac{\partial \mathcal{I}}{\partial S(f)} = 0 \qquad \text{if } S(f) > 0$$
$$\frac{\partial \mathcal{I}}{\partial S(f)} < 0 \qquad \text{if } S(f) = 0$$

Hence for $\{f : S(f) > 0\}$ we have

$$\frac{1/2|H(f)|^2/\sigma^2}{1 + \frac{|H(f)|^2}{\sigma^2} S(f)} = \lambda$$

or

$$\frac{1}{2\lambda} = \frac{\sigma^2}{|H(f)|^2} + S(f), \ f \in \{f : S(f) > 0\}$$

Let $\frac{1}{2\lambda} = \nu$, thus we get the optimal input spectrum $S^*(f)$ as

$$S^*(f) = \nu - \frac{\sigma^2}{|H(f)|^2} \quad f \in \{f : S(f) > 0\}$$

Using the Kuhn-Tucker conditions, we can write

$$S^*(f) = \left(\nu - \frac{\sigma^2}{|H(f)|^2}\right)^+ \quad \forall f \tag{10.4}$$

where $(x)^+ = \begin{cases} x & if x \ge 0 \\ 0 & else \end{cases}$

We can find ν by satisfying the constraint

$$\int S^*(f)df = P, \ i.e. \ \int \left(\nu - \frac{\sigma^2}{|H(f)|^2}\right)^+ df = P$$

Hence the capacity of the ISI channel is,

$$\begin{aligned} C &= \frac{1}{2}\int \log\left(1+S^*(f)\frac{|H(f)|^2}{\sigma^2}\right) df \\ &= \frac{1}{2}\int_{f:\,\left(\nu-\frac{\sigma^2}{|H(f)|^2}\right)>0} \log\left(\frac{\nu|H(f)|^2}{\sigma^2}\right) df \\ &= \frac{1}{2}\int \log\left(1+\left(\frac{\nu|H(f)|^2}{\sigma^2}-1\right)\right) df \end{aligned}$$

The optimal input spectrum given in (10.4) has an interesting interpretation: Consider the channel in figure 10.2

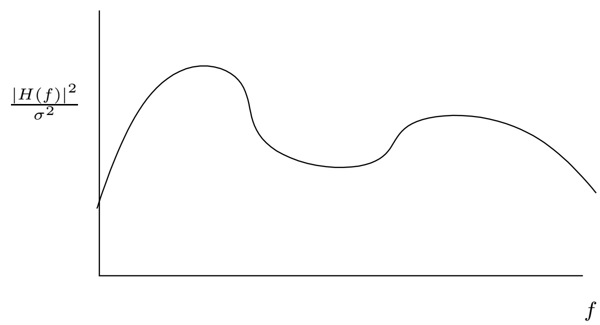


Figure 10.2: Channel spectrum.

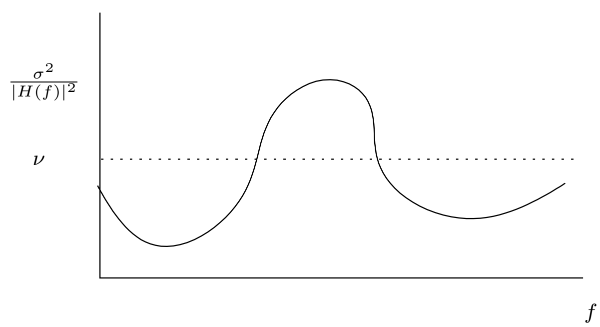


Figure 10.3: Illustration of waterfilling.

The inverted channel is illustrated in Figure 10.3 The relationship in (10.4) implies that $S(f) > 0$ only if $\nu - \frac{\sigma^2}{|H(f)|^2} > 0$. Therefore in Figure 10.2 , we have shown the regions where this occurs for a given ν. The corresponding input spectrum $S(f)$ is shown in Figure 10.4 . An interpretation of this is by thinking of pouring a volume P of water into the crevices of figure 10.3 . The "water" naturally occupies the lowest parts of the spectral valleys and when a total volume P of water is exhausted will have a level ν in the spectral valleys. Therefore, the interpretation (given by Gallager) is called "water-filling" input spectrum for the optimal capacity achieving input distribution.

Notes:

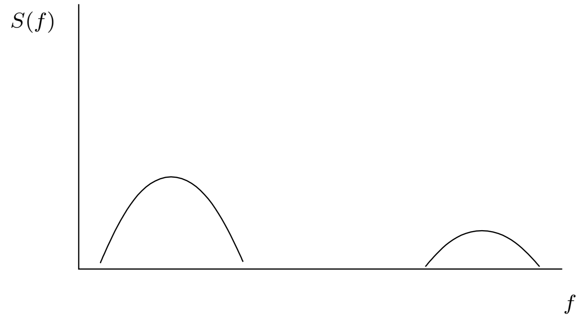


Figure 10.4: Transmit spectrum.

1. The main insight to drawn from the result is that the transmission scheme should focus its power on the "sweet spots" of the channel where gains are the largest. The concavity of the log function (giving diminishing "returns") implies that the power is transmitted only in the best channel response frequency, but descends down by spreading the input power to lower channel gains $|H(f)|^2$.

2. Most of the transmission schemes that we discussed in ISI channels had a "flat" input spectrum, *i.e.* $S_X(f) = P/W$, where W was the transmission bandwidth. Clearly this is not optimal and we will examine the performance of the ISI transmission an receiver structures if we take this into account.

3. A canonical decomposition was through the Fourier basis, creating parallel channels. This decomposition was used in one transmission/reception structure we studied, *i.e.* OFDM. There we did not consider changing the input spectrum, but given this insight we will re-examine that problem

4. The ISI channel capacity pre-suppose that the channel response is *known* at the transmitter. For wire line channels this is realistic since the channel remains constant for a long period and therefore the response can be fed back to the transmitter. However, for wireless channels, such an assumption might be unrealistic. This is a topic outside the scope of this class.

5. The "single carrier" (equalization receiver) structure extensively discussed in the class can also be modified to have a non-uniform input power spectrum. In fact such a structure has been implemented in the V-34 modem for telephone lines.

Next, we compare some of the transmission and reception structures discussed in Chapter 5 and 6 to the capacity of the ISI channel.

10.2 Coded OFDM

Recall from Section 6.2 that in OFDM the cyclic prefix allowed us to relate the input (in the frequency domain) to the output (again in the frequency domain) as

$$Y(l) = P(l)X(l) + Z(l), \quad , l = 0, \dots, N-1$$

where $\{P(l)\}$ was the frequency domain pulse response

$$P(l) = \sum_{n=0}^{\nu} p_n e^{-j\frac{2\pi}{N}ln}, \; l = 0, \dots, N-1$$

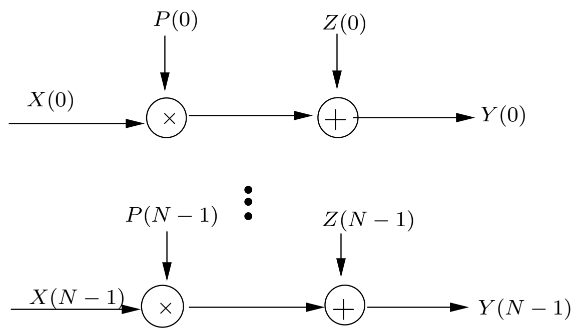


Figure 10.5: Parallel channels in OFDM.

and $\{Z(l)\}$ was the noise (in the frequency domain) and $\{X(l)\}$ was the input in the frequency domain. This relationship can be illustrated as shown in Figure 10.5.

Now, one can concatenate an outer code along with the frequency domain input and code over a sequence of transmission blocks. That is, one can have codes $\{\mathcal{C}_l\}$ for each of the N parallel channels and code such that the inputs come from the corresponding symbols from each block. This is illustrated in figure 10.6 which shows the transformation from a uncoded OFDM system to a coded system.

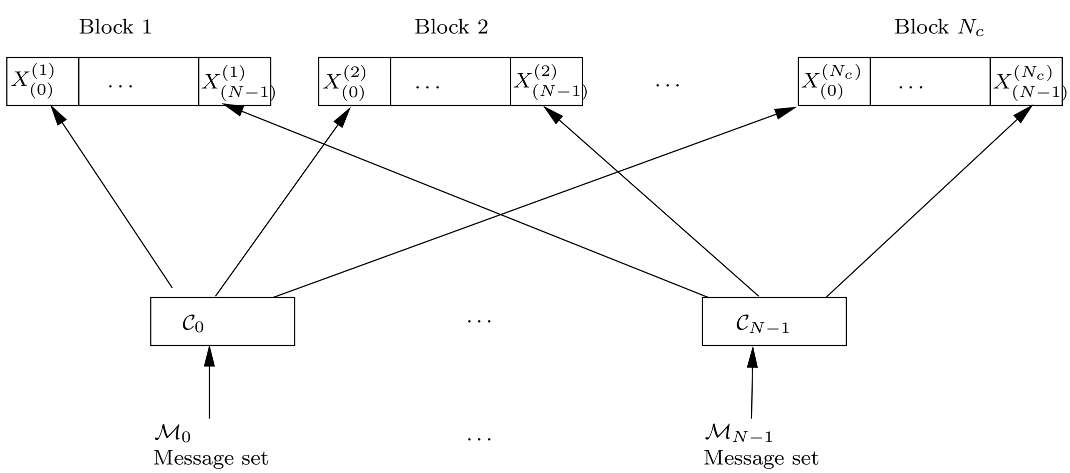


Figure 10.6: A coded OFDM transmissions scheme.

If $|\mathcal{M}_l| = 2^{N_c R_l}$, where the rate on subchannel l is R_l, the code $\mathcal{C}_l$ sends out N_c consecutive symbols which have been coded from the message from set $\mathcal{M}_l$. The symbols from each code form a sequence $\{X^{(k)}(l)\}_{k=0}^{N_c-1}$ of length N_c which is the length of the code. Each symbol is placed in the appropriate frequency sub-carrier of the OFDM. By doing this, for a large enough N_c information theory tells us that the R_l achievable is

$$R_l = \frac{1}{2}\log\left(1 + Q_l\frac{|P(l)|^2}{\sigma^2}\right)$$

where $P(l)$ is the frequency response of the channel and

$$Q_l = \lim_{N_C\to\infty}\frac{1}{N_c}\sum_{k=0}^{N_c-1}|X^{(k)}(l)|^2, \quad l = 0,\dots,N-1$$

is the power assigned to subcarrier l.
Therefore the overall rate achievable for OFDM is

$$R = \sum_{l=0}^{N-1} R_l = \frac{1}{2}\sum_{l=0}^{N-1}\log\left(1 + Q_l\frac{|P(l)|^2}{\sigma^2}\right) \tag{10.5}$$

with the constraint that $\sum_{l=0}^{N-1} Q_l \le P_{tot}$ where P_{tot} is the transmit power. Note that as $N \to \infty$, we get

$$R^* = \lim_{N\to\infty} R = \frac{1}{2}\int\log\left(1 + Q(f)\frac{|P(f)|^2}{\sigma^2}\right)df$$

with $\int Q(f)df \le P_{tot}$.
This looks strikingly similar to the ISI channel capacity expression discussed in (10.3). In fact with the appropriate power allocation we can make the two identical. This gives a hint that OFDM structure is canonical in terms of information theory.

10.2.1 Achievable rate for coded OFDM

Now, let us return to the realm of finite number of subcarriers N. From equation (10.5) the achievable rate of OFDM depends on the power allocated to the various subcarriers, *i.e.* $\{Q_l\}_{l=0}^{N-1}$. We now can maximize the rate by choosing an appropriate power allocation subject to the constraint that $\sum_{l=0}^{N-1} Q_l \le P_{tot}$.

$$maximize\frac{1}{2}\sum_{l=0}^{N-1}\log\left(1 + Q_l\frac{|P(l)|^2}{\sigma^2}\right)$$

$$such\ that\ \sum_{l=0}^{N-1} Q_l \le P_{tot}$$

Clearly this is a problem we have seen (and solved!) before, which led us to the water filling input spectrum. We now solve the discrete version of this problem and give an explicit algorithm that calculates the optimal power distribution $\{Q\}$. As before, let us solve the problem using Kuhn-Tucker conditions. Define the Lagrangian,

$$\mathcal{J} = \frac{1}{2}\sum_{l=0}^{N-1}\log\left(1 + Q_l\frac{|P(l)|^2}{\sigma^2}\right) - \lambda\sum_{l=0}^{N-1} Q_l.$$

This yields,

$$\frac{\partial\mathcal{J}}{\partial Q_l} = \frac{1/2}{1 + Q_l\frac{|P(l)|^2}{\sigma^2}}\frac{|P(l)|^2}{\sigma^2} - \lambda.$$

Using the Kuhn-Tucker conditions, we have for optimality

$$\frac{\partial\mathcal{J}}{\partial Q_l} = 0 \qquad if\ Q_l > 0$$
$$\frac{\partial\mathcal{J}}{\partial Q_l} < 0 \qquad if\ Q_l = 0$$

Thus, for $\{l : Q_l > 0\}$ we get,

$$\frac{1/2}{\frac{\sigma^2}{|P(l)|^2} + Q_l} = \lambda$$

and hence

$$Q_l = \nu - \frac{\sigma^2}{|P(l)|^2} \;\; l \in \{l : Q_l > 0\}$$

where $\nu = 1/2\lambda$. Now using Kuhn-Tucker conditions we can easily verify that the optimal power distribution $\{Q_l^*\}$ can be written as

$$Q_l^* = \left(\nu - \frac{\sigma^2}{|P(l)|^2}\right)^+ \quad \forall l \in \{0, \dots, N-1\} \tag{10.6}$$

where $(x)^+ = \begin{cases} x & if x \geq 0 \\ 0 & else \end{cases}$

and ν is chosen such that $\sum_{l=0}^{N-1} \left(\nu - \frac{\sigma^2}{|P(l)|^2}\right)^+ \leq P_{tot}$.

The "waterfilling" algorithm computes ν and therefore gives us a method to calculate the optimal power distribution. The algorithm is derived in Section 9.2.2.

10.2.2 Waterfilling algorithm

Step 1

If $Q_l > 0 \; \forall l$, then finding ν becomes easy

$$\sum_{l=0}^{N-1} \left[\nu - \frac{\sigma^2}{|P(l)|^2}\right] = P_{tot}$$

This yields,

$$N\nu = P_{tot} + \sum_{l=0}^{N-1} \frac{\sigma^2}{|P(l)|^2}$$

$$\text{or } \nu = \frac{1}{N}\left[P_{tot} + \frac{1}{N}\sum_{l=0}^{N-1} \frac{\sigma^2}{|P(l)|^2}\right] \tag{10.7}$$

However, for such a choice of ν, we have

$$\begin{aligned} Q_l &= \nu - \frac{\sigma^2}{|P(l)|^2} \\ &= \frac{1}{N}P_{tot} + \frac{1}{N}\sum_{l=0}^{N-1} \frac{\sigma^2}{|P(l)|^2} - \frac{\sigma^2}{|P(l)|^2} \end{aligned} \tag{10.8}$$

There is no guarantee that the Q_l that arises from a choice of ν given in equation (10.7) would be non-negative in (10.8) in general.

Step 2

If we know the set of "active" subchannels $\mathcal{A} = \{l : Q_l > 0\}$, then again finding ν becomes easy.

$$\sum_{l \in \mathcal{A}} \left(\nu - \frac{\sigma^2}{|P(l)|^2} \right) = P_{tot}$$

This yields,

$$\begin{aligned} \nu &= \frac{1}{|\mathcal{A}|} P_{tot} + \frac{1}{|\mathcal{A}|} \sum_{l \in \mathcal{A}} \frac{\sigma^2}{|P(l)|^2} \\ \text{and } Q_l &= \begin{cases} \frac{1}{|\mathcal{A}|} P_{tot} + \frac{1}{|\mathcal{A}|} \sum_{k \in \mathcal{A}} \frac{\sigma^2}{|P(k)|^2} - \frac{\sigma^2}{|P(l)^2|} & l \in \mathcal{A} \\ 0 & otherwise \end{cases} \end{aligned}$$

However, one does not know the active set $\mathcal{A}$.

Step 3

In order to find the correct active set we can start by assuming $\mathcal{A} = \{0, \ldots, N-1\}$, *i.e.* the entire set of subcarriers. Then from (10.7) we can find the appropriate ν. But (10.8) would tell us which Q_l are negative. If there is $Q_l < 0$, then we drop the "worst" channel from the active list *i.e.*,

$$l_1 = \arg\max_{l} \frac{\sigma^2}{|P(l)|^2}$$

and consider the active list as $\mathcal{A} = \{0, \ldots, N-1\} \setminus l_1$, where the notation means the set difference. Given this order preference in the choice of active subchannels we reduce the number of candidate active sets from $N!$ to N by ordering $\left\{ \frac{\sigma^2}{|P(l)|^2} \right\}$ in increasing order. We can index the candidate active sets by choice (in increasing order) of the values $\left\{ \frac{\sigma^2}{|P(l)|^2} \right\}$. By doing this one can find the optimal active set. The main step in this is the observation that if a particular subchannel l' is active, then all subchannels such that

$$\frac{\sigma^2}{|P(l)|^2} \leq \frac{\sigma^2}{|P(l')|^2}$$

are also active. This can be seen from the relationship in (10.6) and we next give a formal statement and proof of this claim.

Claim 10.2.1. *If the optimal active set $\mathcal{A}^*$ contains subchannel l', then it must also contain elements $\tilde{l}$ when*

$$\frac{\sigma^2}{|P(\tilde{l})|^2} < \frac{\sigma^2}{|P(l')|^2} \tag{10.9}$$

Proof: Suppose, $\mathcal{A}^*$ is the optimal active set and it contains element l' and does *not* contain an element $\tilde{l}$ which is such that

$$\frac{\sigma^2}{|P(\tilde{l})|^2} < \frac{\sigma^2}{|P(l')|^2}$$

Now, we prove the assertion by contradiction. Since $l' \in \mathcal{A}^*$ and $\tilde{l} \notin \mathcal{A}^* \Rightarrow Q_{l'} > 0$, $Q_{\tilde{l}} = 0$. And the optimal rate is

$$R^* = \frac{1}{2} \sum_{l \in \mathcal{A}^*} \log \left(1 + Q_l \frac{|P(l)|^2}{\sigma^2} \right) \tag{10.10}$$

Now consider a different power allocation $\{\hat{Q}_l\}$ where all the allocation is the same except we choose for small enough $\varepsilon > 0$,

$$\hat{Q}_{l'} = Q_{l'} - \varepsilon \text{ and } \hat{Q}_{\tilde{l}} = \varepsilon > 0$$

For this allocation

$$\begin{aligned}
\hat{R} &= \frac{1}{2}\sum_{l\in\mathcal{A}^*\backslash l'} \log\left(1 + Q_l\frac{|P(l)|^2}{\sigma^2}\right) \\
&\quad + \frac{1}{2}\log\left(1 + (Q_{l'} - \varepsilon)\frac{|P(l')|^2}{\sigma^2}\right) \\
&\quad + \frac{1}{2}\log\left(1 + \varepsilon\frac{|P(\tilde{l})|^2}{\sigma^2}\right) \qquad (10.11)
\end{aligned}$$

Using (10.10) and (10.11) we get

$$\begin{aligned}
R^* - \hat{R} &= \frac{1}{2}\log\left(1 + Q_{l'}\frac{|P(l')|^2}{\sigma^2}\right) \\
&\quad - \frac{1}{2}\log\left(1 + (Q_{l'} - \varepsilon)\frac{|P(l')|^2}{\sigma^2}\right) \\
&\quad - \frac{1}{2}\log\left(1 + \varepsilon\frac{|P(\tilde{l})|^2}{\sigma^2}\right) \\
&\overset{(a)}{\leq} \frac{1}{2}\log\left(1 + Q_{l'}\frac{|P(\tilde{l})|^2}{\sigma^2}\right) \\
&\quad - \frac{1}{2}\log\left(1 + (Q_{l'} - \varepsilon)\frac{|P(l')|^2}{\sigma^2}\right) \\
&\quad - \frac{1}{2}\log\left(1 + \varepsilon\frac{|P(l')|^2}{\sigma^2}\right) \qquad (10.12)
\end{aligned}$$

where (a) follows because log is an increasing function and $\frac{|P(\tilde{l})|^2}{\sigma^2} > \frac{|P(l')|^2}{\sigma^2}$ due to assumption (10.10). Continuing with (10.12) we get

$$\begin{aligned}
R^* - \hat{R} &\leq \frac{1}{2}\log\left(1 + Q_{l'}\frac{|P(l')|^2}{\sigma^2}\right) \\
&\quad - \frac{1}{2}\log\left(1 + \varepsilon\frac{|P(l')|^2}{\sigma^2}\right) - \frac{1}{2}\log\left(1 + (Q_{l'} - \varepsilon)\frac{|P(l')|^2}{\sigma^2}\right) \\
&= \frac{1}{2}\log\left(\frac{\sigma^2 + Q_{l'}|P(l')|^2}{\sigma^2 + \varepsilon|P(l')|^2}\right) - \frac{1}{2}\log\left(1 + (Q_{l'} - \varepsilon)\frac{|P(l')|^2}{\sigma^2}\right) \\
&= \frac{1}{2}\log\left(1 + \frac{(Q_{l'} - \varepsilon)|P(l')|^2}{\sigma^2 + \varepsilon|P(l')|^2}\right) - \frac{1}{2}\log\left(1 + (Q_{l'} - \varepsilon)\frac{|P(l')|^2}{\sigma^2}\right) \\
&< 0
\end{aligned}$$

Hence $R^* < \hat{R}$ and therefore $\mathcal{A}^*$ cannot be the optimal active set assignment. Therefore we have a contradiction and hence the claim is proved.

□

Therefore, we need to consider only N candidate active sets instead of $N!$ and therefore giving a $O(N)$ steps water filling algorithm.

10.2.3 Algorithm Analysis

1. Sort $\left\{\frac{\sigma^2}{|P(l)|^2}\right\}$ in ascending order, *i.e.*

$$\frac{\sigma^2}{|P(l_1)|^2} \leq \frac{\sigma^2}{|P(l_2)|^2} \leq \ldots \leq \frac{\sigma^2}{|P(l_N)|^2}$$

2. Let $\mathcal{A} = \{l_1, l_2, \ldots, l_N\}$, $p = N$

3. Compute

$$\nu = \frac{1}{|\mathcal{A}|} P_{tot} + \frac{1}{|\mathcal{A}|} \sum_{k \in \mathcal{A}} \frac{\sigma^2}{|P(k)|^2}$$

4. If $\exists Q_k < 0$, then let $\mathcal{A} = \mathcal{A} \setminus l_p$ and $p = p - 1$. Goto step 3.
   Else optimal power allocation found.

Notes

1. This algorithm terminates in at most N steps.

2. The sorting takes $O(N \log N)$ complexity and the computation of ν takes $O(N)$ complexity. Hence we have $O(N \log N)$ algorithm for computing the optimal power allocation.

3. We devised this algorithm based on claim 10.2.1 where we showed that the ordering of $\left\{\frac{\sigma^2}{|P(l)|^2}\right\}$ determined the active set. Therefore we choose the complete set as $\mathcal{A}$ first and then eliminate the worst channel, one-by-one till we get the correct $\mathcal{A}^*$.

4. Ties in this algorithm can be resolved arbitrarily since the final rate does not get affected.

5. The algorithm does not take into account particular alphabets/constellations one might use for the transmission. Other algorithms that utilize this have been found. They are outside the scope of this class.

10.3 An information-theoretic approach to MMSE-DFE

Fact 1 If $(\mathbf{x}, \mathbf{y})$ are jointly Gaussian then $\mathbb{P}_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})$ is also Gaussian with mean $\mathbb{E}[\mathbf{x}|\mathbf{y}]$ and covariance $\mathbb{E}\left[\{\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}]\}\{\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{y}]\}^*\right] = \mathbf{R}^{\perp}_{\mathbf{X}|\mathbf{Y}}$ which is the error covariance of a linear estimator of $\mathbf{x}$ from $\mathbf{y}$. ■

Fact 2 For a Gaussian random vector $\mathbf{x} \sim \mathbb{C}\eta(0, \mathbf{R}_{xx})$ the entropy is

$$H_x = \log\left[(\pi e)^N |\mathbf{R}_{xx}|\right]$$

where N is the dimension of $\mathbf{x}$ and $|\mathbf{R}_{xx}|$ is the determinant of $\mathbf{R}_{xx}$

Fact 3 If $(\mathbf{x}, \mathbf{y})$ are jointly Gaussian, then the conditional entropy $H_{\mathbf{X}|\mathbf{Y}}$ is

$$H_{\mathbf{X}|\mathbf{Y}} = \log\left[(\pi e)^N |\mathbf{R}^{\perp}_{\mathbf{X}|\mathbf{Y}}|\right]$$

where $\mathbf{R}^{\perp}_{\mathbf{X}|\mathbf{Y}}$ is the error covariance of linear estimator of $\mathbf{x}$ from $\mathbf{y}$.

Now,

$$\mathbf{R}^{\perp}_{\mathbf{x}|\mathbf{y}} = \mathbf{R}_{xx} - \mathbf{R}_{xy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}$$

where

$$\mathbf{R}_{xy} = \mathbb{E}\left[\mathbf{x}\mathbf{y}^*\right], \; \mathbf{R}_{yx} = \mathbb{E}\left[\mathbf{y}\mathbf{x}^*\right], \; \mathbf{R}_{yy} = \mathbb{E}\left[\mathbf{y}\mathbf{y}^*\right]$$

For a stationary Gaussian process, the chain rule of entropy allows computation of its entropy based on prediction. For a sequence $\mathbf{x} = \{x_k, x_{k-1}, \ldots, x_0\}$,

$$\begin{aligned} H_{\mathbf{x}} &= H_{x_k|[x_{k-1},\ldots,x_0]} + H_{x_{k-1}|[x_{k-2},\ldots,x_0]} + \ldots \\ &\quad + H_{x_1|x_0} + H_{x_0} \\ &= \sum_{n=0}^{k} H_{x_n|[x_{n-1},\ldots,x_0]} \end{aligned}$$

where we have defined $H_{x_0|x_{-1}} \stackrel{\Delta}{=} H_{x_0}$.

Definition 10.3.1. *For a stationary process $\{x_k\}$, the entropy rate $H_{x(D)}$ is,*

$$H_{x(D)} = \lim_{k\to\infty} \frac{1}{k+1} H([x_k, \ldots, x_0]) = \lim_{k\to\infty} \frac{1}{k+1} \sum_{n=0}^{k} H(x_n|[x_{n-1}, \ldots, x_0])$$

For a stationary sequence this quantity converges to

$$H_{x(D)} = \lim_{k\to\infty} H(x_k|x_{k-1}, \ldots)$$

This is related to the estimation error associated with predicting x_k from its past, *i.e.*

$$H_{x(D)} = \log\left[\pi e \sigma^2_{x|past}\right]$$

where $\sigma^2_{x|past}$ is the MMSE error of predicting x_k (the "present") from the entire history of the sequence. Given input $\{x_k\}$ and output $\{y_k\}$

$$I(x(D); y(D)) = \lim_{k\to\infty} \frac{1}{k+1} I(\mathbf{x}_0^k, \mathbf{y}_0^k)$$

where $\mathbf{x}_0^k = \{x_0, \ldots, x_l\}$, $\mathbf{y}_0^k = \{y_0, \ldots, y_k\}$ and $I(\mathbf{x}; \mathbf{y})$ is the *mutual information* between $\mathbf{x}$ and $\mathbf{y}$. For an inter-symbol interference (ISI) channel, we have seen from before that

$$I(x(D); y(D)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left[\frac{S_y(w)}{S_u(w)}\right] dw$$

where $S_y(w)$ and $S_z(w)$ are the power spectral densities of the processes $\{y_k\}$ and the noise $\{z_k\}$ respectively. Now

$$\begin{aligned} I(x(D); y(D)) &= H_{x(D)} - H_{x(D)|y(D)} \\ &= H_{x_k|x_{k-1}\ldots} - H_{x_k|y_{-\infty}^{\infty}, x_{k-1},\ldots} \end{aligned} \tag{10.13}$$

Therefore the mutual information can be written as,

$$I(x(D); y(D)) = I(x_k; y_{-\infty}^{\infty}|x_{-\infty}^{k-1})$$

10.3.1 Relationship of mutual information to MMSE-DFE

From 10.13 we see that

$$\begin{aligned} I(x(D); y(D)) &= H_{x_k|x_{-\infty}^{k-1}} - H_{x_k|y_{-\infty}^{\infty}, x_{-\infty}^{k-1}} \\ &= \log(\pi e)\sigma^2_{x_k|x_{-\infty}^{k-1}} - \log(\pi e)\sigma^2_{x_k|y_{-\infty}^{\infty}, x_{-\infty}^{k-1}} \end{aligned} \tag{10.14}$$

But,

$$\begin{aligned} \sigma^2_{x_k|x_{-\infty}^{k-1}} &= \gamma_x = \text{ MMSE associated with predicting } x_k \text{ from its past} \\ \sigma^2_{x_k|y_{-\infty}^{\infty}, x_{-\infty}^{k-1}} &= \sigma^2_{MMSE-DFE} = \text{ The variance of error in perfect past decisions, DFE.} \end{aligned}$$

Inserting this into 10.14 we get,

$$\begin{aligned} I(x(D); y(D)) &= \log \frac{\gamma_x}{\sigma^2_{MMSE-DFE}} \\ &= \log SNR_{MMSE-DFE} \\ &= \log\left[1 + SNR_{MMSE-DFE,U}\right] \end{aligned}$$

This results in what is knows as the "CDEF result" (after the result of Cioffi-Dudevoir-Eyuboglu-Forney done in 1995).

Lemma 10.3.1 (CDEF result)**.** *The unbiaised SNR of a MMSE-DFE is related to the mutual information for a linear ISI channel with additive Gaussian noise in exactly the same formula as the SNR of an ISI free channel is related to the mutual information of that channel. Assuming that the input and output are jointly (stationary) Gaussian and the MMSE-DFE exists.*

10.3.2 Consequences of CDEF result

1. This result seems to show that with the perfect past decisions assumption, the MMSE-DFE is a canonical structure, *i.e.* it can achieve the capacity of the ISI channel.

2. In the development of the MMSE-DFE, we assumed i.i.d. inputs, which meant a "flat" input spectrum $S_x(\omega)$. The development we had on the capacity of the ISI channels indicates that this might not be optimal. Hence the modification we need to make the MMSE-DFE canonical, is input spectral shaping.

3. There are several other caveats and we will next do a more careful derivation of the result to illuminate these caveats.

Caveats of CDEF results

To do this we establish first a lemma.

Lemma 10.3.2. *Let* $\mathbf{x} = \{x_k\}$, $\mathbf{y} = \{y_k\}$ *be jointly Gaussian zero mean random processes, then*

$$I(\mathbf{x}; \mathbf{y}) = I(\mathbf{x}; \pi_{\mathbf{y}}(x))$$

where $\pi_{\mathbf{y}}(x)$ *is the orthogonal projection operator of* $\mathbf{y}$ *onto* $\mathbf{x}$, i.e., *from Gaussian processes,*

$$\pi_{\mathbf{y}}(x) = \mathbb{E}\left[\mathbf{x}|\mathbf{y}\right]$$

Recall from linear prediction, of process $\{x_k\}$, *i.e.*, predicting x_k from its infinite past,

$$\hat{x_k} = \sum_{m=1}^{\infty} a_m x_{k-m} \Rightarrow \hat{X}(D) = A(D)X(D)$$

with $A(D)$ being strictly causal. $A(D) = 1 - A'(D)$ and we found that

$$A'(D) = \frac{1}{L(D)} \text{ where}$$

$$S_x(D) = \gamma_x L(D) L^*(D^{-*})$$

where $L(D)$ is monic, minimum and stable.
Now, using the chain rule of mutual information,

$$I(x_k, y_{-\infty}^{\infty}, x_{-\infty}^{k-1}) = I(x_k; x_{-\infty}^{k-1}) + I(x_k; y_{-\infty}^{\infty} | x_{-\infty}^{k-1})$$

Hence, using (10.14) we get

$$I(X(D); Y(D)) = I(x_k; y_{-\infty}^{\infty}, x_{-\infty}^{k-1}) - I(x_k; x_{-\infty}^{k-1})$$

Therefore, unless we have a white input, *i.e.* $I(x_k; x_{-\infty}^{k-1}) = 0$, the two quantities $I(x_k; y_{-\infty}^{\infty}, x_{-\infty}^{k-1})$ and $I(X(D); Y(D))$ are distinct and in fact,

$$I(X(D); Y(D)) \le I(x_k; y_{-\infty}^{\infty}, x_{-\infty}^{k-1})$$

with equality iff the input is white.
However, as we have seen in the waterfilling solution to achieving ISI channel capacity, the optimal sequence $\{x_k\}$ need not to be white and therefore the above quantities are distinct. It can be shown that

$$\begin{aligned} I(x_k; y_{-\infty}^{\infty}, x_{-\infty}^{k-1}) &= I(x_k; r_k) + I(e_k; e_{-\infty}^{k-1}) \\ &= I(x_k; r_k + \hat{e}_k) \end{aligned} \tag{10.15}$$

where

$$r_k = \mathbb{E}\left[x_k | y_{-\infty}^{\infty}\right], e_k = x_k - \tilde{x}_k$$

Pictorially this is depicted in Figure 10.7.
In Figure 10.7, we use, $I(e_k; e_{-\infty}^{k-1}) = I(e_k; \hat{e}_k)$, where $\hat{e}_k = \mathbb{E}\left[e_k | e_{-\infty}^{k-1}\right]$, by using the lemma. Moreover for

$$e_k = x_k - r_k,$$

it has a spectral factorization

$$S_e(D) = \gamma_e L_e(D) L_e^*(D^{-*})$$

yielding a linear predictor filter,

$$A'_e(D) = \frac{1}{L_e(D)}$$

$$\begin{aligned} \hat{E}(D) &= [1 - A'_e(D)]E(D) = [1 - A'_e(D)][X(D) - R(D)] \\ &= X(D) - R(D) - A'_e(D)X(D) + A'_e(D)R(D) \\ \Rightarrow R(D) + \hat{E}(D) &= X(D)[1 - A'_e(D)] + A'(D)R(D) \end{aligned}$$

Using this we can modify figure (10.7) as,

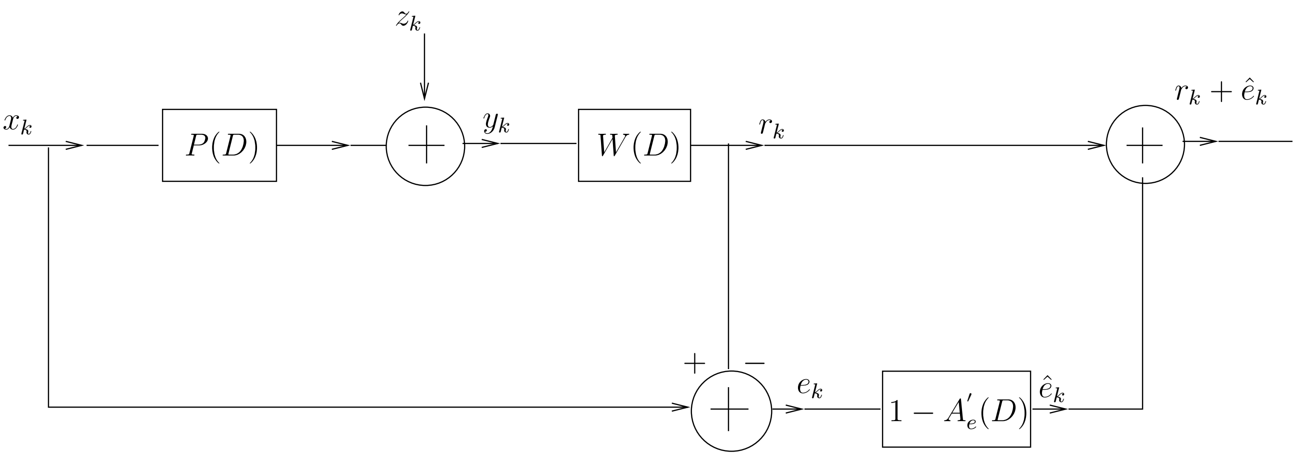


Figure 10.7: Depiction of MMSE-DFE with perfect decision feedback.

Now figure 10.8 almost looks like the MMSE-DFE with perfect past decisions, since $A'_e(D) - 1$ is strictly causal. However, from (10.15) it is clear that it represents $I(x_k; y_{-\infty}^{\infty}, x_{-\infty}^{k-1})$ which in general is larger than the ISI channel capacity. Therefore we really need to look at

$$I(x_k; y_{-\infty}^{\infty}, x_{-\infty}^{k-1}) - I(x_k; x_{-\infty}^{k-1}) = I(X(D); Y(D))$$

to find the capacity of the ISI channel and relate it to the DFE case. This caveat shows that there is no artificial boost and that one has to look at the channel symbol-by-symbol wise rather than sequence-wise.

Main message:

1. If one interprets the CDEF result correctly, we see that the DFE combined with input spectral shaping and sophisticated coding can actually be a canonical transmission/reception scheme.

2. There are several caveats, one is that of perfect decision feedback.

3. The other is to view the channel symbol-wise and not sequence-wise.

Let us relate the $SNR_{MMSE-DFE}$ to the CDEF result. Now, if we use Salz formula for $\sigma^2_{MMSE-DFE}$,

$$\sigma^2_{MMSE-DFE} = \frac{N_0/2}{||p||^2} \exp\left(-\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \ln\left(Q(e^{-j\omega T}) + \frac{1}{SNR_{MFB}} \right) d\omega \right)$$

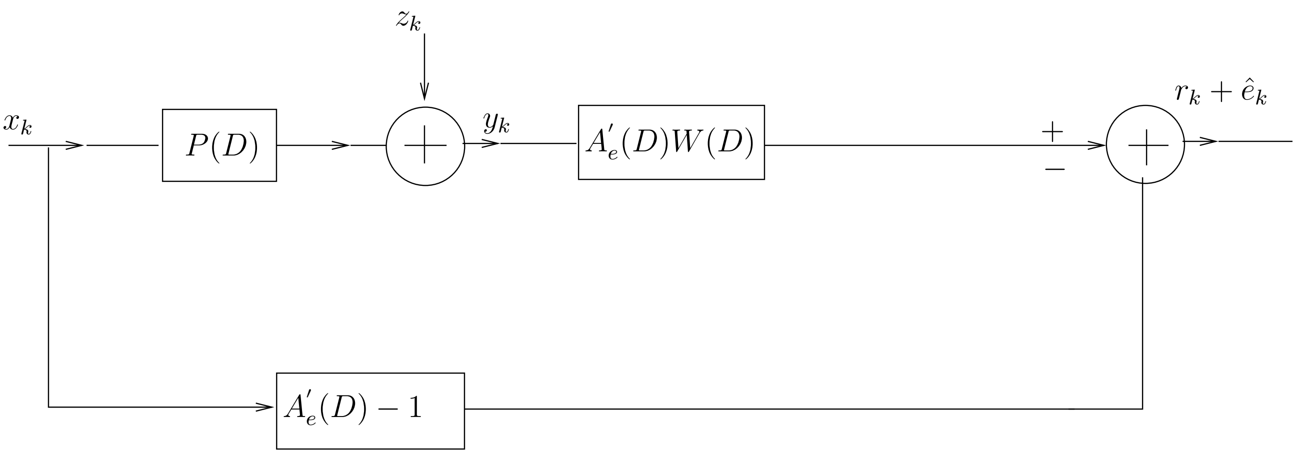


Figure 10.8: Modified form of Figure 10.7.

Using the Salz formula in the CDEF result we get

$$
\begin{aligned}
I(X(D);Y(D)) &= \log\left[\frac{\gamma_x}{\frac{N_0/2}{||p||^2}}\exp\left(\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\log\left\{Q(e^{-j\omega T}) + \frac{1}{SNR_{MFB}}\right\}\right)d\omega\right] \\
&= \log\left(\frac{||p||^2\gamma_x}{N_0/2}\right) + \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\log\left[Q(e^{-j\omega T}) + \frac{1}{SNR_{MFB}}\right]d\omega \\
&= \log SNR_{MFB} + \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\log\frac{1}{SNR_{MFB}}d\omega \\
&\quad + \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\log\left[Q(e^{-j\omega T})SNR_{MFB} + 1\right]d\omega \\
&= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\log\left[1 + Q(e^{-j\omega T})SNR_{MFB}\right]d\omega
\end{aligned}
$$

This result for i.i.d. inputs and we can improve the rates by input spectral shaping. The V.34 modem does this with precoding to get close to the predicted channel capacity.

10.4 Problems

Problem 10.1

Consider a set of parallel independent AWGN channels:

1. Show that the mutual information for the set of channels is the sum of the mutual information quantities for the set.

2. If the set of parallel channels has a total energy constraint that is equal to the sum of the energy constraints, what energy $\mathcal{E}_n$, $n = 1 \cdots N$ should be allocated to each of the channels to maximize the mutual information. You may presume the subchannel gains are given as g_n (so that the individual SNRs would be then $\mathcal{E}_n g_n$).

3. Find the overall SNR for a single AWGN that is equivalent to the set of channels in terms of mutual information.

Problem 10.2

In this problem we study the water filling algorithm for the oversampled version of the coded OFDM transmission. Recall that for an over sampling factor L the parallel channel relationship is given by

$$\mathbf{Y}_k(l) = \mathbf{D}_l \mathbf{X}_k(l) + \mathbf{Z}_k(l), \quad l = 0, \ldots, N-1,$$

where $\mathbf{Y}_k(l), \mathbf{D}_l, \mathbf{Z}_k(l) \in \mathbb{C}^L$. For more detail refer to section 6.2.2 of the reader. In this case the rate for l^{th} parallel channel is given by

$$R_l = \frac{1}{2} \ln \left(1 + Q_l \frac{||\mathbf{D}_l||^2}{\sigma^2} \right),$$

where

$$Q_l = \lim_{N_c \to \infty} \frac{1}{N_c} \sum_{k=1}^{N_c} |X^{(k)}(l)|^2, \quad l = 0, 1, \ldots, N-1$$

is the power assigned to subcarrier l. Now consider a particular case with $L = 2$ and channel memory $\nu = 1$ with

$$[\mathbf{p_0} \quad \mathbf{p_1}] = \begin{bmatrix} 1 & 1.81 \\ 0 & 1 \end{bmatrix}$$

and let the number of subcarriers be 4.

1. Find the solution to the maximization problem

$$\text{maximize} \quad \sum_{l=0}^{3} R_l$$

   such that

$$\sum_{l=0}^{3} Q_l \leq P$$

   with $P = 0.01$ and $\sigma^2 = 0.1$.

2. Perform the water filling algorithm and point out the active sets in each step. Find the values of $\{Q_l\}$.

Part V

Appendix

Appendix A

Mathematical Preliminaries

A.1 The Q function

The Q function is defined as:

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{\xi^2}{2}} d\xi.$$

Hence, if $Z \sim \mathcal{N}(0,1)$ (meaning that Z is a Normally distributed zero-mean random variable of unit variance) then $Pr\{Z \geq x\} = Q(x)$.

If $Z \sim \mathcal{N}(m, \sigma^2)$, then the probability $Pr\{Z \geq x\}$ can be written using the Q function by noticing that $\{Z \geq x\}$ is equivalent to $\{\frac{Z-m}{\sigma} \geq \frac{x-m}{\sigma}\}$. Hence $Pr\{Z \geq x\} = Q(\frac{x-m}{\sigma})$.

We now describe some of the key properties of $Q(x)$.

(a) If $Z \sim \mathcal{N}(0,1)$, $F_Z(z) = Pr\{Z \leq z\} = 1 - Q(z)$.

(b) $Q(0) = 1/2$, $Q(-\infty) = 1$, $Q(\infty) = 0$.

(c) $Q(-x) + Q(x) = 1$.

(d) $\frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{\alpha^2}{2}} (1 - \frac{1}{\alpha^2}) < Q(\alpha) < \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{\alpha^2}{2}}$, $\alpha > 0$.

(e) An alternative expression with fixed integration limits is $Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta$. It holds for $x \geq 0$.

(f) $Q(\alpha) \leq \frac{1}{2} e^{-\frac{\alpha^2}{2}}$, $\alpha \geq 0$.

A.2 Fourier Transform

A.2.1 Definition

$$
\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt \\
h(t) &= \int_{-\infty}^{\infty} H(f)e^{2\pi jft}df
\end{aligned}
$$

A.2.2 Properties of the Fourier Transform

$$
\begin{aligned}
x(t) * y(t) &\Longleftrightarrow X(f)Y(f) \\
h(t)e^{j2\pi f_0 t} &\Longleftrightarrow H(f-f_0) \\
h^*(-t) &\Longleftrightarrow H^*(f) \\
h(t-s) &\Longleftrightarrow H(f)e^{-2\pi jfs} \\
h(t/a) &\Longleftrightarrow aH(fa) \\
sinc(t) = \frac{\sin(\pi t)}{\pi t} &\Longleftrightarrow rect(f) = \begin{cases} 1, & |f| \leq \frac{1}{2} \\ 0, & |f| > \frac{1}{2} \end{cases} \\
\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &\Longleftrightarrow H(f)G(f) \\
\int_{-\infty}^{\infty} h(\tau)g^*(\tau-t)d\tau &\Longleftrightarrow H(f)G^*(f) \\
\int_{-\infty}^{\infty} h(t)g^*(t)dt &\Longleftrightarrow \int_{-\infty}^{\infty} H(f)G^*(f)df
\end{aligned}
$$

A.2.3 Basic Properties of the sinc Function

Using the above relations we get:

$$
\begin{aligned}
sinc(\frac{t}{\tau}) &\Longleftrightarrow \begin{cases} \tau, & |f| \leq \frac{1}{2\tau} \\ 0, & |f| > \frac{1}{2\tau} \end{cases} \\
sinc(\frac{t}{\tau} - n) &\Longleftrightarrow \begin{cases} \tau e^{-2\pi jn\tau f}, & |f| \leq \frac{1}{2\tau} \\ 0, & |f| > \frac{1}{2\tau} \end{cases} \\
\int_{-\infty}^{\infty} sinc(\frac{t}{\tau} - n)sinc(\frac{t}{\tau} - m)dt &= \int_{-\frac{1}{2\tau}}^{\frac{1}{2\tau}} \tau^2 e^{-2\pi j(n-m)\tau f}df = \begin{cases} 0, & m \neq n \\ \tau, & m = n \end{cases}
\end{aligned}
$$

From the last equality we conclude that $sinc(\frac{t}{\tau})$ is orthogonal to all of its shifts (by multiples of τ). Further, we see that the functions $\sqrt{\frac{1}{\tau}}sinc(\frac{t}{\tau} - n), n \in \mathbb{Z}$, form an orthonormal set. One can also show that this set is complete for the class of square integrable functions which are low-pass limited to $\frac{1}{2\tau}$.

A.3 Z-Transform

A.3.1 Definition

Assume we have a discrete time (real or complex valued) signal x_n, $n \in \mathbb{Z}$. Its associated z-transform, call it $X(z)$ (if it exists), is defined by

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

The region of convergence, known as the ROC, is important to understand because it defines the region where the z-transform exists. The ROC for a given $x[n]$ is defined as the range of z for which the z-transform converges. By the Cauchy criterion, a power series $\sum_{k=0}^{\infty} u(k)$ converges if $\lim_{k\to\infty} |u(k)|^{\frac{1}{k}} < 1$. One can write

$$\sum_{n=-\infty}^{+\infty} x(n) z^{-n} = \sum_{n=1}^{+\infty} x(-n) z^{n} + \sum_{n=0}^{+\infty} x(n) z^{-n}$$

and it follows by the Cauchy criterion that the first series converges if $|z| < \lim_{k\to\infty} \frac{1}{|x(-k)|^{\frac{1}{k}}} = R_{x+}$ and the second converges if $|z| > \lim_{k\to\infty} |x(k)|^{\frac{1}{k}} = R_{x-}$ Then the region of convergence is an annular region such that $R_{x-} < |z| < R_{x+}$

A.3.2 Basic Properties

$$\begin{aligned}
x^*_{-n} &\iff X^*(1/z^*) \\
x_{n-m} &\iff X(z) z^{-m} \\
\sum_k x_k y_{n-k} &\iff X(z) Y(z) \\
\sum_k x_k y^*_{n-k} &\iff X(z) Y^*(1/z^*)
\end{aligned}$$

We say that a sequence x_n is *causal* if $x_n = 0$ for $n < 0$ and we say that it is *anticausal* if $x_n = 0$ for $n > 0$. For a causal sequence the ROC is of the form $|z| > R$ whereas for an anticausal it is of the form $|z| < R$. We say that a sequence is *stable* if $\sum_n |x_n| < \infty$. The ROC of a stable sequence must contain the unit circle. If $X(z)$, the z-transform of x_n, is rational then it implies that for a stable and causal system all the poles of $X(z)$ must be within the unit circle. Finally, we say that a sequence x_n with rational z-transform $X(z)$ is *minimum phase*, if all its poles and zeros are within the unit circle. Such a sequence has the property that for all $N \geq 0$ it maximizes the quantity $\sum_{n=0}^{N} |x_n|^2$ over all sequences which have the same $|H(z)|$.

A.4 Energy and power constraints

The signal $x(t)$ is said to have finite energy if

$$\mathcal{E}_x \triangleq \int |x(t)|^2 dt < \infty$$

and it is said to have finite power if

$$\mathcal{P}_x \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} |x(t)|^2 dt < \infty$$

For signals of the first type, we define the autocorrelation function of $x(t)$ as

$$\phi_x(\tau) \triangleq \int x(t)x(t-\tau)^* dt$$

For signals of the second type, we define the time-averaged autocorrelation function

$$\phi_x(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)x(t-\tau)^* dt$$

Let $\mathcal{F}[.]$ denote the Fourier transform operator, such that $X(f) = \mathcal{F}[x] = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$. For a finite-energy signals $x(t)$, $|X(f)|^2 = \mathcal{F}[\phi_x]$ is called *energy spectral density* (ESD). In fact because of Parseval identity,

$$\int |X(f)|^2 df = \phi_x(0) = \mathcal{E}_x$$

For finite-power signals $x(t)$, we define the *power spectral density* (PSD) $S_x(f) \triangleq \mathcal{F}[\phi_x]$. In fact, $\int S_x(f)df = \phi_x(0) = \mathcal{P}_x$.

The output of a LTI system with impulse response $h(t)$ to the input $x(t)$ is given by the convolution integral

$$y(t) = h(t) * x(t) \triangleq \int h(\tau)x(t-\tau)d\tau$$

In the frequency domain, we have $Y(f) = H(f)X(f)$, where $H(f) = \mathcal{F}(h)$ is the system *transfer function.* The ESD (resp. PSD) of $y(t)$ and $x(t)$ are related by: $|Y(f)|^2 = |H(f)|^2|X(f)|^2$ (resp. $S_y(f) = |H(f)|^2 S_x(f)$), where $|H(f)|^2$ is the system energy (resp. power) transfer function. In the time domain, we have

$$\phi_y(\tau) = \phi_h(\tau) * \phi_x(\tau)$$

A.5 Random Processes

A random process $x(t)$ can be seen either as a sequence of random variables $x(t_1), x(t_2), ..., x(t_n)$ indexed by the "time" index $t = t_1, t_2, ...$, or as a collection of signals $x(t;\omega)$, where ω is a random experiment taking on values in a certain event space Ω. The full statistical characterization of a random process $x(t)$ is given by the collection of all joint probability cumulative distribution functions (cdf)

$$\Pr(x(t_1) \le x_1, x(t_2) \le x_2, ..., x(t_n) \le x_n)$$

for all $n = 1, 2, ...$ and for all instant $t_1, t_2, ..., t_n$.

Complex random variables and processes are characterized by the joint statistics of its real and imaginary parts. For example, a random variable $X = X_1 + jX_2$ is characterized by the joint cdf $\Pr(X_1 \le x_1, X_2 \le x_2)$. A complex random variable is said to be *circularly-symmetric* if its real and imaginary parts satisfy

$$cov(X_1, X_2) = 0, \;\; var(X_1) = var(X_2)$$

The first and second order statistics of $x(t)$ are given by its mean

$$\mu_x(t) = E[x(t)]$$

and by its autocorrelation function

$$\phi_x(t_1, t_2) = E[x(t_1)x(t_2)^*]$$

For two random processes $x(t)$ and $y(t)$, defined on a joint probability space, we define the *cross-correlation* function

$$\phi_{xy}(t_1, t_2) = E[x(t_1)y(t_2)^*]$$

A.6 Wide sense stationary processes

A random process $x(t)$ is said to be wide-sense stationary (WSS) if

(a) $\mu_x(t) = \mu_x$ is constant with t

(b) $\phi_x(t_1, t_2)$ depends only on the difference $\tau = t_1 - t_2$ (we can use the notation $\phi_x(\tau) \triangleq \phi_x(t+\tau, t)$).

Two random processes $x(t)$ and $y(t)$ are said to be jointly WSS if both $x(t)$ and $y(t)$ are individually WSS and if their cross-correlation function $\phi_{xy}(t_1, t_2)$ depends only on the difference $t_1 - t_2$.

For WSS processes, we have

$$S_x(f) = \mathcal{F}[\phi_x] = \int \phi_x(t) e^{-j2\pi f t} dt$$

and for jointly WSS processes, the cross-spectrum is given by $S_{xy}(f) = \mathcal{F}[\phi_{xy}(\tau)]$.

The output of a LTI system with impulse response $h(t)$ to the WSS input $x(t)$ is the WSS process given by

$$y(t) = h(t) * x(t) \triangleq \int h(\tau) x(t-\tau) d\tau$$

The two processes $x(t)$ and $y(t)$ are jointly WSS. The mean and autocorrelation of $y(t)$ and the cross-correlation between $x(t)$ and $y(t)$ are given by

$$\mu_y = \mu_x \int h(t) dt$$

$$\phi_y(\tau) = \phi_h(\tau) * \phi_x(\tau)$$

$$\phi_{xy}(\tau) = h(-\tau)^* * \phi_x(\tau)$$

In the frequency domain we have

$$\mu_y = \mu_x H(0)$$

$$S_y(f) = |H(f)|^2 S_x(f)$$

$$S_{xy}(f) = H^*(f) S_x(f)$$

Since $\phi_{yx}(\tau) = \phi_{xy}(-\tau)^*$, we have $S_{yx}(f) = S_{xy}(f)^*$, that yields $S_{yx}(f) = H(f) S_x(f)$, since $S_x(f)$ is real.

A.7 Gram-Schmidt orthonormalisation

Let V be an inner product space and let $A = a_1, ..., a_m$ be a set of elements of V. The following Gram-Schmidt procedure then allows us to find an *orthonormal basis* for A. Let this basis be $\psi_1, ..., \psi_n$, $n \leq m$, so that

$$a_i = \sum_{j=1}^{n} \langle a_i, \psi_j \rangle \psi_j, \quad i \in [n]$$

This basis is recursively defined by (ignoring cases of dependent vectors)

$$\begin{aligned}
\psi_1 &= \frac{a_1}{\sqrt{\langle a_1, a_1\rangle}} \\
\psi_2 &= \frac{a_2 - \langle a_2, \psi_1\rangle\psi_1}{\sqrt{\langle (a_2 - \langle a_2, \psi_1\rangle\psi_1), (a_2 - \langle a_2, \psi_1\rangle\psi_1)\rangle}} \\
\vdots & \quad \ldots \\
\psi_n &= \frac{a_m - \sum_{j=1}^{m-1}\langle a_m, \psi_j\rangle\psi_j}{\sqrt{\langle (a_m - \sum_{j=1}^{m-1}\langle a_m, \psi_j\rangle\psi_j), (a_m - \sum_{j=1}^{m-1}\langle a_m, \psi_j\rangle\psi_j)\rangle}}
\end{aligned}$$

In general, the basis obtained by the above algorithm depends on the order in which the elements a_i are considered. Different ordering yield different bases for the same vector space.

A.8 The Sampling Theorem

Let $s(t)$ be a function in $\mathcal{L}^2$ that is lowpass limited to B. Then $s(t)$ is specified by its values at a sequence of points spaced at $T = \frac{1}{2B}$ intervals by the interpolation formula:

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT)\,\mathrm{sinc}(\frac{t}{T} - n)$$

where $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

The sinc pulse does not have unit energy. Hence we define (its normalized version) $\psi(t) = \frac{1}{\sqrt{T}}\,\mathrm{sinc}\,(\frac{t}{T})$. The set $\{\psi(t - iT)\}_{i=-\infty}^{\infty}$ forms an orthonormal set. Hence we can write:

$$s(t) = \sum_{i=-\infty}^{\infty} s_i\psi(t - iT)$$

where $s_i = s(nT)\sqrt{T}$. This highlights the way the sampling theorem should be seen, namely as a particular instance of an orthonormal expansion. In this expansion the basis is formed by time translated sinc pulses. Implicit in the sampling theorem is the fact that the set $\{\psi(t - iT)\}_{i=-\infty}^{\infty}$ is a complete orthonormal basis for the set of waveforms that are lowpass limited to $B = \frac{1}{2T}$.

A.9 Nyquist Criterion

We are looking for functions $\psi(t - T)$ (like the sinc function constructed above) with the property: $\int_{-\infty}^{\infty} \psi(t - nT)\psi^*(t)dt = \delta_n$

We now look for the condition under which a real-valued function $\psi(t)$ ensures that $\psi(t), \psi(t-T), \psi(t-2T), \ldots$ forms an orthonormal sequence.

Define

$$g(f) = \sum_{k\in\mathbb{N}} \psi_{\mathcal{F}}(f + \frac{k}{T})\psi^*_{\mathcal{F}}(f + \frac{k}{T}).$$

where $\psi_{\mathcal{F}}(f) = \mathcal{F}\psi(t)$ Now

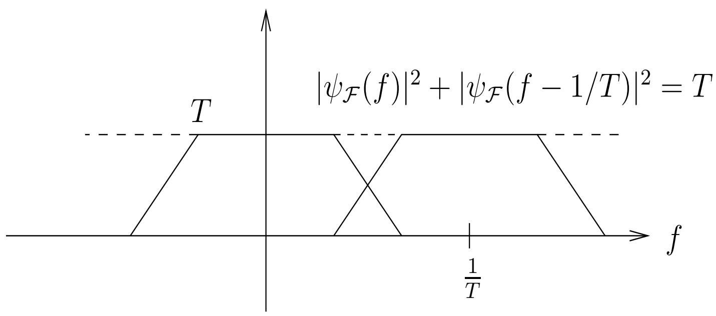
$$\begin{aligned}
\delta_n &= \int_{-\infty}^{\infty} \psi_{\mathcal{F}}(t-nT)\psi_{\mathcal{F}}^*(t)dt \\
&\overset{\text{(Parseval)}}{=} \int_{-\infty}^{\infty} \psi_{\mathcal{F}}(f)\psi_{\mathcal{F}}^*(f)e^{-j2\pi nTf}df \\
&= \sum_{-\infty}^{\infty} \int_{(2m-1)/2T}^{(2m+1)/2T} \psi_{\mathcal{F}}(f)\psi_{\mathcal{F}}^*(f)e^{-j2\pi nTf}df \\
&= \sum_{-\infty}^{\infty} \int_{-1/2T}^{1/2T} \psi_{\mathcal{F}}(f+m/T)\psi_{\mathcal{F}}^*(f+m/T)e^{-j2\pi nTf}df \\
&= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} g(f)e^{-j2\pi nTf}df.
\end{aligned}$$

The last expression is T times the n-th Fourier series coefficient of $g(f)$. Since only the coefficient with $n=0$ is nonzero, the function $g(f)$ must be constant. Specifically, $g(f) \equiv T, f \in [-\frac{1}{2T}, \frac{1}{2T}]$.

Then, one can state :

A waveform $\psi(t)$ is orthonormal to each shift $\psi(t-nT)$ if and only if

$$\sum_{k=-\infty}^{\infty} |\psi_{\mathcal{F}}(f+\frac{k}{T})|^2 = T \quad \text{for } f \in [-\frac{1}{2T}, \frac{1}{2T}]$$



A.10 Choleski Decomposition

Given a Hermitian positive definite matrix A, the Cholesky decomposition is a diagonal matrix D and an upper triangular matrix U with ones on the main diagonal such that

$$A = U^*DU.$$

A.11 Problems

Problem A.1

Prove the following bounds on Q-function for $\alpha > 0$:

$$\frac{1}{\sqrt{2\pi}\alpha}e^{-\frac{\alpha^2}{2}}(1-\frac{1}{\alpha^2}) < Q(\alpha) < \frac{1}{\sqrt{2\pi}\alpha}e^{-\frac{\alpha^2}{2}}.$$

Hint: $e^{-\frac{y^2}{2}} = e^{-\frac{y^2}{2}} y \frac{1}{y}$ and integrate by parts.

Problem A.2

Prove the following properties of fourier transform:

- frequency shift - $h(t)e^{j2\pi f_0 t} \Longleftrightarrow H(f - f_0)$.
- time shift - $h(t - s) \Longleftrightarrow H(f)e^{-2\pi jfs}$.
- lateral inversion - $h^*(-t) \Longleftrightarrow H^*(f)$.
- time scaling - $h\left(\frac{t}{a}\right) \Longleftrightarrow |a|H(fa)$.

Problem A.3

Given $0 < a < b$, find the temporal sequence $x(n)$ of

$$X(z) = \frac{2 - (a + b)z^{-1}}{(1 - az^{-1})(1 - bz^{-1})},$$

when

(a) the ROC is $|z| > b$.

(b) the ROC is $a < |z| < b$.

Problem A.4

(a) A random process $\{Z(t)\}$ is given by

$$Z(t) = \sin(2\pi f_0 t + \Theta),$$

where Θ is uniformaly distributed on $[-\pi, \pi]$. Find its power spectral density.

(b) Let $\{X(t)\}$ be a WSS process with autocorrelation function $\phi_X(\tau) = e^{-|\tau|}$. Find $\mathbb{E}\left[(X(0) + X(2))^2\right]$.

(c) Let $W(t) = Y_{\lfloor t \rfloor}$, where $\{Y_i\}_{-\infty}^{\infty}$ are independent zero-mean, unit-variance Gaussian random variables. Is $\{W(t)\}$ a WSS process?

Problem A.5

A zero-mean WSS process $x(t)$ with autocorrelation function $\phi_X(\tau) = e^{-|\tau|}$ is passed through a LTI filter with impulse response $h(t) = e^{-t}$. Show that $x(t)$ and $y(t)$ are jointly WSS. Find $\phi_Y(\tau)$ and $\phi_{XY}(\tau)$.

Problem A.6

In this exercise we continue our review of what happens when stationary stochastic processes are *filtered*. Let $X(t)$ and $U(t)$ denote two stochastic processes and let $Y(t)$ and $V(t)$ be the result of passing $X(t)$ respectively $U(t)$ through linear time invariant filters with impulse response $h(t)$ and $g(t)$, respectively. For any pair (X, U) of stochastic processes define the cross-correlation as

$$\mathcal{R}_{XU}(t_1, t_2) = \mathbb{E}[X(t_1)U^*(t_2)],$$

We say that the pair (X, U) is jointly wide sense stationary if each of them is wide sense stationary and if $\mathcal{R}_{XU}(t_1, t_2)$ is a function of the time difference only. In this case we define a cross-power spectrum as the Fourier transform of the cross-correlation function.

Show that if (X, U) are jointly wide sense stationary then so are (Y, V) and that

$$\mathcal{S}_{YV}(f) = \mathcal{S}_{XU}(f)H(f)G^*(f).$$

Problem A.7

Show that the cross-correlation function $\mathcal{R}_{XU}(\tau)$ has symmetry

$$\mathcal{R}_{XU}(\tau) = \mathcal{R}^*_{UX}(-\tau)$$

Problem A.8

(a) Let X_r and X_i be statistically independent zero-mean Gaussian random variables with identical variances. Show that a (rotational) transformation of the form

$$Y_r + jY_i = (X_r + jX_i)e^{j\phi}$$

results in another pair (Y_r, Y_i) of Gaussian random variables that have the same joint PDF as the pair (X_r, X_i).

(b) Note that

$$\begin{bmatrix} Y_r \\ Y_i \end{bmatrix} = A \begin{bmatrix} X_r \\ X_i \end{bmatrix}$$

where A is a 2×2 matrix. As a generalization of the transformation considered in (1), what property must the linear transformation A satisfy if the PDFs for X and Y, where $Y = AX$, $X = (X_1, X_2, \cdots, X_n)$ and $Y = (Y_1, Y_2, \cdots, Y_n)$ are identical? Here also we assume that $(X_1, \cdots, X_n)$ are zero-mean statistically independent Gaussian random variables with same variance.

Problem A.9

[Transformation of Gaussian Random Variables] Let $Z = (Z_1, \ldots, Z_n)$ denote a jointly Gaussian vector with independent components with zero mean and each with variance σ^2, i.e., we have

$$f_Z(z) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{\|z\|^2}{2\sigma^2}}$$

Let $\{\psi_1, \ldots, \psi_n\}$ be any basis for $\mathbb{R}^n$, i.e., an orthonormal set and let $W = (W_1, \ldots, W_n)$ denote a random vector whose components are the projections of Z onto this basis, i.e, $W_i = \langle Z, \psi_i \rangle$. Show that W has the same distribution as Z, i.e., W is a jointly Gaussian vector with independent components with zero mean and each with variance σ^2.

Problem A.10

Let $Z(t)$ be a real-valued Gaussian process with double-sided power spectral density equal to $\frac{N_0}{2}$. Let $\psi_1(t)$ and $\psi_2(t)$ be two orthonormal functions and for $k = 0, 1$ define the random variables $Z_k = \int_{-\infty}^{\infty} Z(t)\psi_k(t)dt$. What is the distribution of (Z_1, Z_2)?