Tensegrity Systems

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Course Outline

- What is Tensegrity?

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- What is Tensegrity?
- Tensegrity in
    - Nature
    - Art
    - Engineering

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- Statics
  - Minimal mass compressive structures
  - Minimal mass bending structures
  - Tensegrity Fractals

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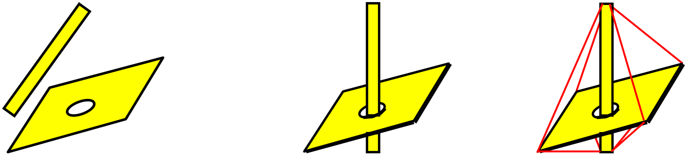
- Dynamics
- Control

What is a Tensegrity System?

A *configuration* of rigid bodies forms a **tensegrity configuration** if the given configuration can be stabilized by some set of tensile members connected between the rigid bodies.



Not a Tensegrity Configuration   Tensegrity Configuration   Tensegrity System

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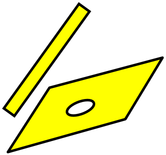
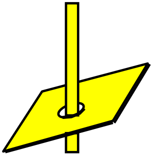
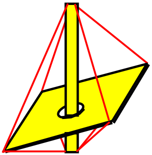


Not a Tensegrity Configuration



Tensegrity Configuration



Tensegrity System

A **tensegrity system** is composed of a **tensegrity configuration** of rigid bodies and any given set of strings connecting the rigid bodies.

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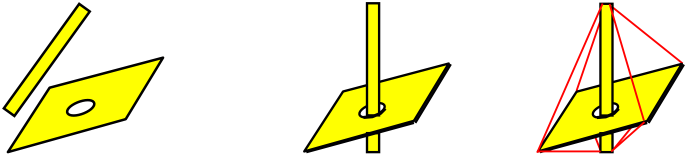


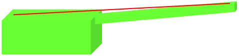
Not a Tensegrity Configuration   Tensegrity Configuration   Tensegrity System

A **tensegrity system** is composed of a **tensegrity configuration** of rigid bodies and any given set of strings connecting the rigid bodies.

Note that a **tensegrity configuration** without strings forms an *unstable tensegrity system*. (An insufficient set of strings might be added, even though a stabilizing set exists).

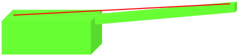
Class k tensegrity systems

class 1 tensegrity system class 2 tensegrity system class 3 tensegrity system



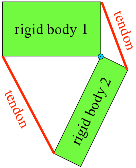
Class k tensegrity systems

class 1 tensegrity system



class 2 tensegrity system

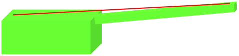
2 rigid bodies
2 tensile elements



class 3 tensegrity system

Class k tensegrity systems

class 1 tensegrity system



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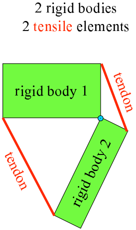

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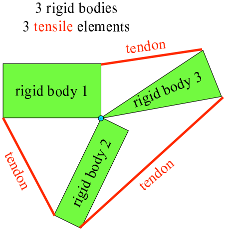
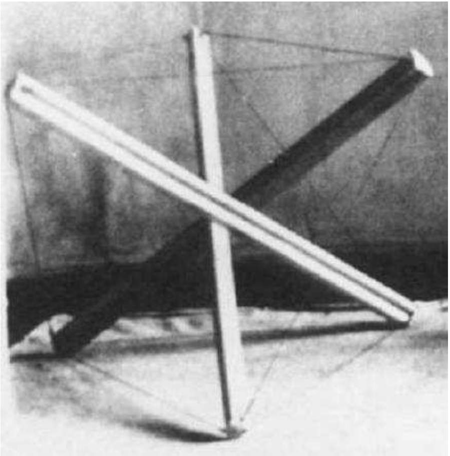
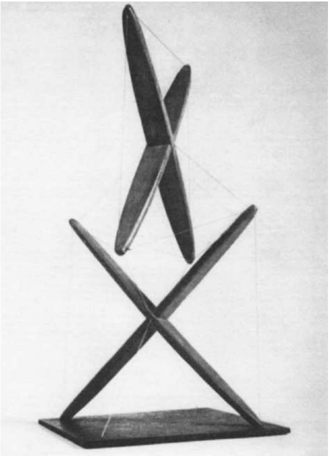


Tensegrity in art



Ioganson's 3-bar, 9-string stable structure, 1921.



Snelson's X-piece, 1948

- system of muscles, bones, tendons for animal locomotion
- molecular structure of spider fiber

tensegrity systems in Nature

- system of muscles, bones, tendons for animal locomotion
- molecular structure of spider fiber
- cells in biological material

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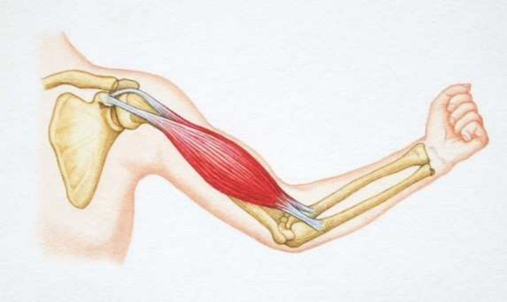
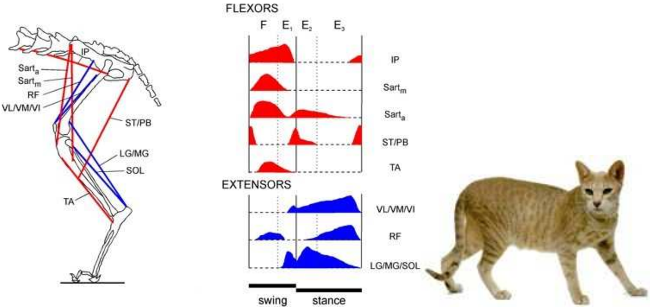


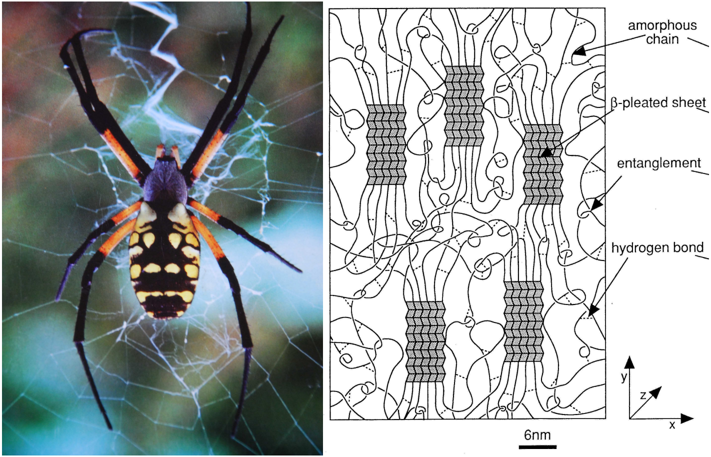
The elbow can be thought of as a class 3 tensegrity joint, the shoulder as a class 2 tensegrity joint, and the foot as a class 2 tensegrity joint. The total system would be called a class 3 tensegrity system

Cat locomotion



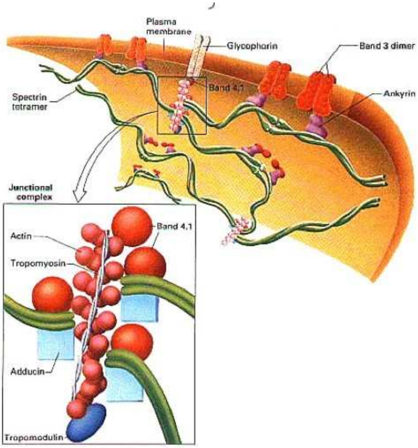
The flexor tendons (red) and the extensor tendons (blue) of a cat's hind legs.
The plot shows the time profile of the forces in each tendon during a walk

The dragline silk of a Nephila Clavipes



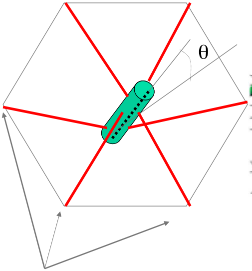
The molecular structure of nature's strongest fiber. A class 1 tensegrity model. The rigid bodies are the β-pleated sheets, and the tensile members are the amorphous strands that connect to the β-pleated sheets

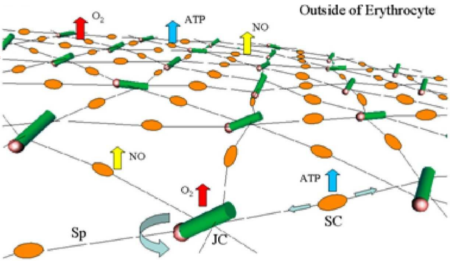
Tensegrity in red blood cells



The network of junctional complexes underneath the red blood cell membrane. The protofilament (33,000 in each red blood cell) is the rigid body and the spectrin dimer are the tensile members. Each spectrin is stapled to the lipid bilayer

Tensegrity network in red blood cells





A single junctional complex within the network of 33,000 junctional complexes underneath the red blood cell membrane. Each of the 6 binding sites where the strings connect to the rigid body are known. The actin protofilament (rod) has a radius of 4.5 nm and a length of 37 nm.

Fullerenes and Carbon Nanotubes

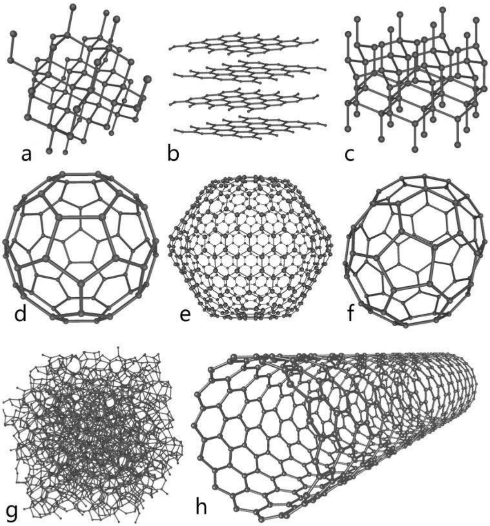


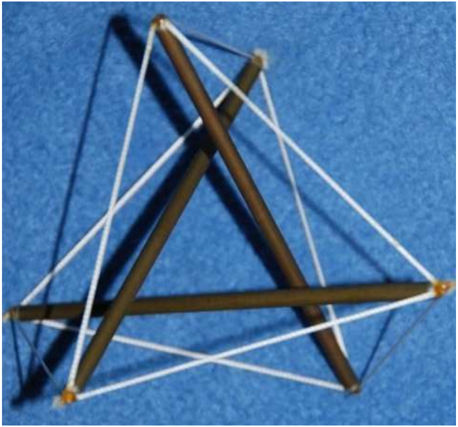
Image by Michael Strck

a) Diamond
b) Graphite
c) Lonsdaleite
d) C_{60} Fullerene
e) C_{540} Fullerene
f) C_{70} Fullerene
g) Amorphous carbon
h) Carbon nanotube

Tensegrity in Engineering: A Bicycle

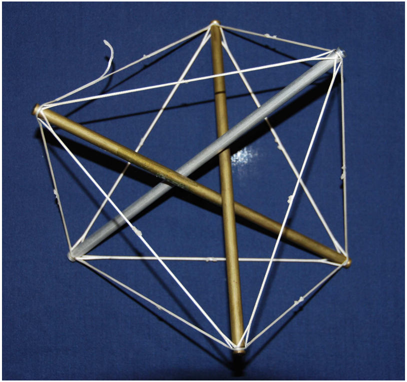
A class 1 tensegrity structure
rigid body one: rim
rigid body two: hub

A class 1 tensegrity wing. One rigid body has the shape of the airfoil, and the other body has the shape of a long rod (the spar). None of these rigid bodies touch each other.



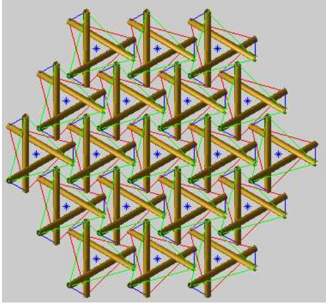
Top view of A *regular minimal tensegrity prism* for $p = 3$.

Regular non-minimal tensegrity prism

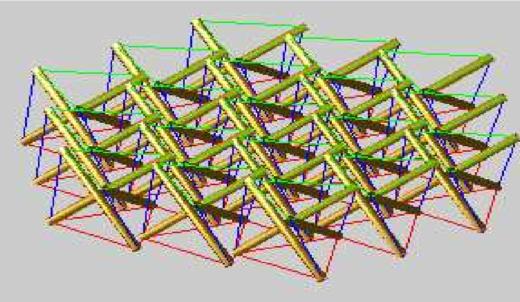


A stiff regular non-minimal tensegrity prism. The minimal number of strings that can stabilize is 9. This structure has 12 strings.

class 1 tensegrity plate



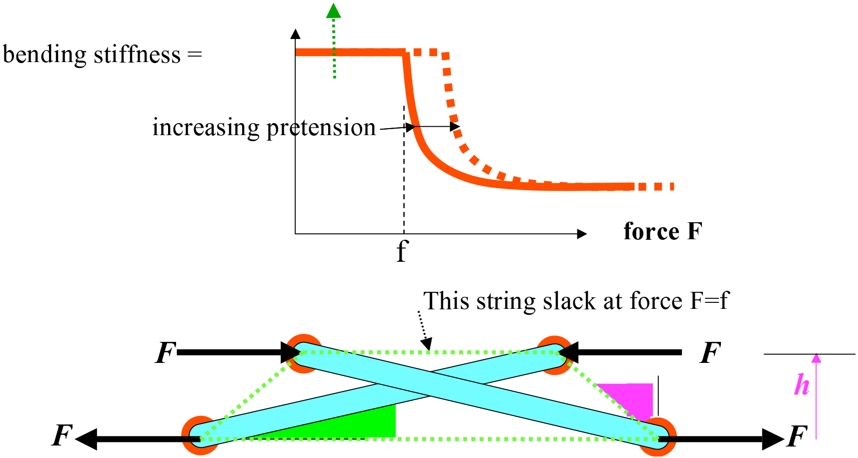
Top view



Perspective view

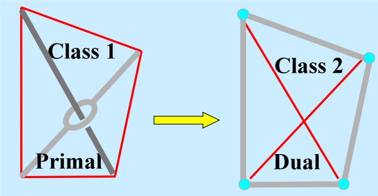
A plate constructed from *regular minimal tensegrity prisms*

Robustness from prestress

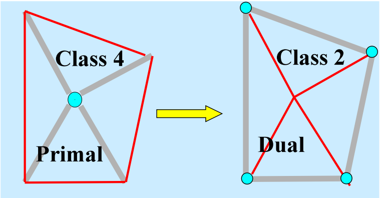


A class 1 tensegrity structure in bending. The bending stiffness is dominated by geometry, and the robustness to uncertainty in external moments is dictated by prestress. The stiffness drops when a string goes slack

Tensegrity duals



Stable primal and 3D unstable dual



3D unstable primal and dual

Any system of bars and strings (*primal*) has a *dual* obtained by replacing bars by strings and vice-versa

Optimal Tensegrity for Bending Loads

What is Optimal Structure for Bending?

Statement of the Problem to be solved:
Given:
an upper bound q on the number of elements to be used in construction of the structure (this will be called the complexity q of the system), and
an aspect ratio (the ratio of the length of the structure and the foundation dimension attached to the structure)
Find:
the structure that has minimal mass, subject to material yield constraints

Define circles of radii r_ℓ. Define members of lengths p_ℓ. Define Michell spiral:

$$r_{\ell+1} = ar_\ell, \qquad p_\ell = cr_\ell, \qquad \ell = 0, 1, 2, \cdots, q, \tag{1}$$

where $a > 0$ and $c > 0$.

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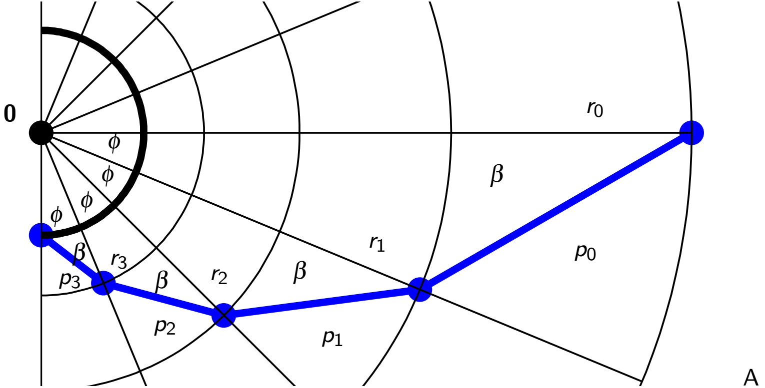
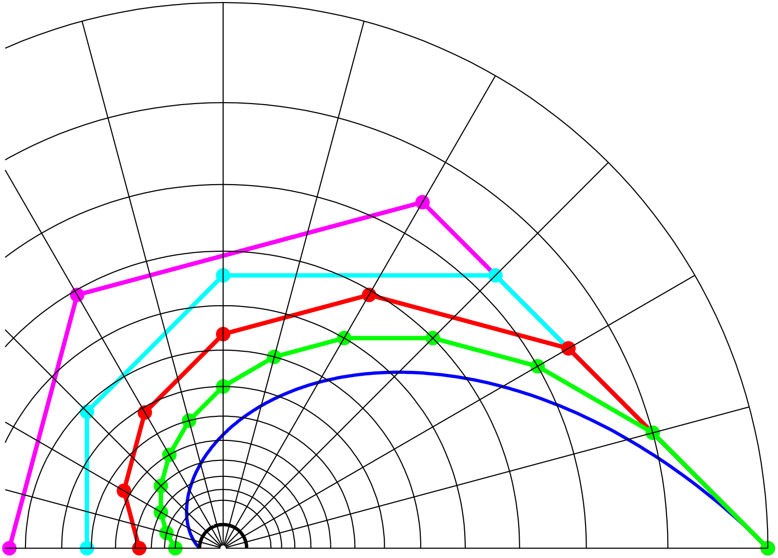


Michell Spiral of Order 4 ($\phi = \pi/16$, $\beta = \pi/6$), where a and c are

$$a = \frac{\sin\beta}{\sin(\beta+\phi)}, \qquad c = \frac{\sin\phi}{\sin(\beta+\phi)}$$

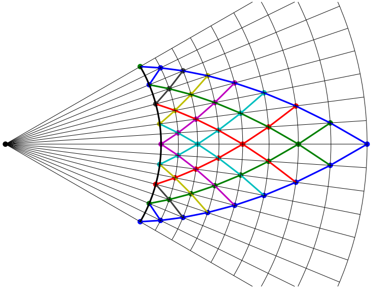
More Michell spirals



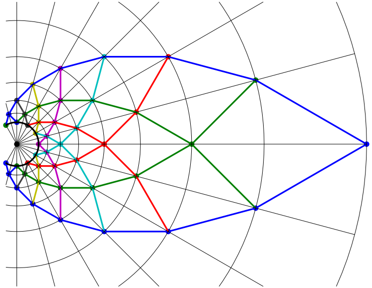
Discrete Michell Spirals of Order 3, 4, 6, 12 and ∞ (continuous) ($q\,\phi = \pi$, $\beta = \pi/4$)

Michell Topologies of order 8

A Michell Topology of order k is formed
by the set of all Michell spirals of order $\leq k$, and their conjugates (mirror images).

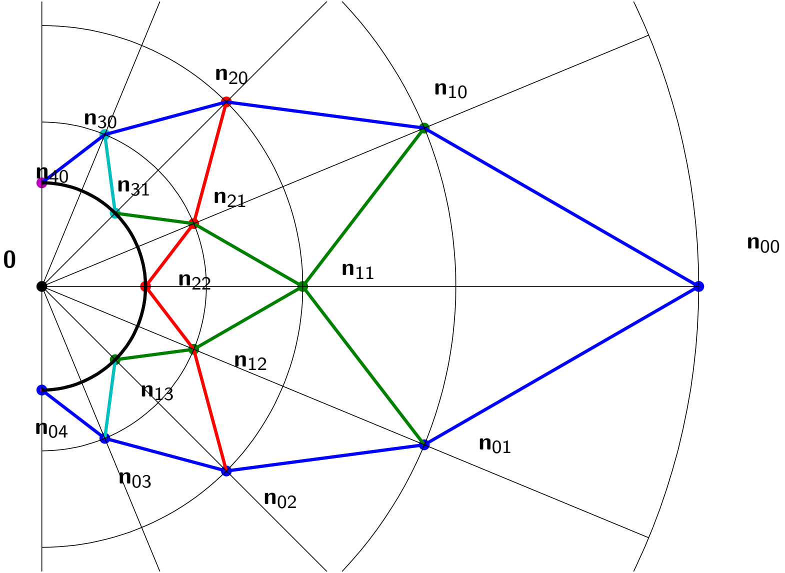


$(\phi = \pi/48,\ \beta = \pi/6)$



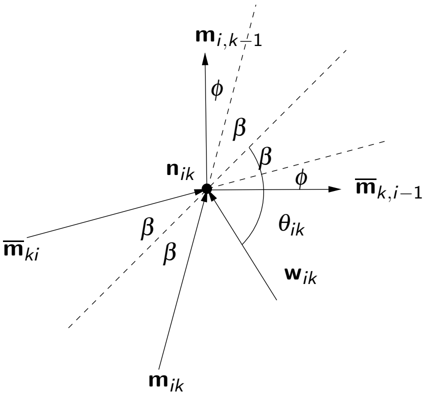
$(\phi = \pi/12,\ \beta = \pi/6)$

numbering convention



Michell Topology of Order 4 ($\phi = \pi/16$, $\beta = \pi/6$)

Summing forces at node n_{ik}



Forces along directions $\mathbf{m}_{ik}$, $\overline{\mathbf{m}}_{ki}$, $\mathbf{m}_{i,k-1}$, $\overline{\mathbf{m}}_{k,i-1}$, respectively have magnitudes, f_{ik}, t_{ki}, $f_{i,k-1}$, $t_{k,i-1}$.

Force equilibrium

$$f_{ik}\frac{\mathbf{m}_{ik}}{\|\mathbf{m}_{ik}\|}+t_{ki}\frac{\overline{\mathbf{m}}_{ki}}{\|\mathbf{m}_{ki}\|}-f_{i,k-1}\frac{\mathbf{m}_{i,k-1}}{\|\mathbf{m}_{i,k-1}\|}-t_{k,i-1}\frac{\overline{\mathbf{m}}_{k,i-1}}{\|\mathbf{m}_{k,i-1}\|}+\mathbf{w}_{ik}=0.$$

which reduces to:

$$\begin{pmatrix} t_{ki} \\ f_{ik} \end{pmatrix} p_{i+k} = \mathbf{\Omega} \begin{pmatrix} t_{k,i-1} \\ f_{i,k-1} \end{pmatrix} p_{i+k-1} + \mathbf{\Phi}_{ik} w_{ik},$$

with “initial” conditions,

$$\begin{pmatrix} t_{00} \\ f_{00} \end{pmatrix} p_0 = \mathbf{\Phi}_{00} w_{00},$$

where,

$$\mathbf{\Phi}_{ik} = \frac{p_{i+k}}{\sin(2\beta)} \begin{bmatrix} \sin(\theta_{ik}-\beta) \\ -\sin(\theta_{ik}+\beta) \end{bmatrix},$$

$$\mathbf{\Omega} = \frac{1}{2}\begin{bmatrix} g & -h \\ -h & g \end{bmatrix}, \qquad g = 1+\frac{\tan\beta}{\tan(\beta+\phi)}, \quad h = \frac{\sin\phi}{\cos\beta\sin(\beta+\phi)}.$$

and $g+h=2$, a useful fact.

Vector form of equilibrium

Define vector $\mathbf{x}_\alpha \in \mathbb{R}^{2(\alpha+1)}$ by the forces in all members that lie within the radii r_α and $r_{\alpha+1}$. That is,

$$\mathbf{x}_0 = \begin{bmatrix} t_{00} \\ f_{00} \end{bmatrix} p_0, \quad \mathbf{x}_1 = \begin{bmatrix} t_{01} \\ f_{10} \\ t_{10} \\ f_{01} \end{bmatrix} p_1, \quad \mathbf{x}_2 = \begin{bmatrix} t_{02} \\ f_{20} \\ t_{11} \\ f_{11} \\ t_{20} \\ f_{02} \end{bmatrix} p_2, \quad \mathbf{x}_3 = \begin{bmatrix} t_{03} \\ f_{30} \\ t_{12} \\ f_{21} \\ t_{21} \\ f_{12} \\ t_{30} \\ f_{03} \end{bmatrix} p_3 \quad \mathbf{x}_\alpha = \begin{bmatrix} t_{0\alpha} \\ f_{\alpha 0} \\ t_{1,\alpha-1} \\ f_{\alpha-1,1} \\ \vdots \\ t_{i,\alpha-i} \\ f_{\alpha-i,i} \\ \vdots \\ t_{\alpha,0} \\ f_{0,\alpha} \end{bmatrix} p_\alpha.$$

The normalized forces in all members between radii r_0 and r_1, between radii r_1 and r_2, between radii r_2 and r_3, and between radii r_3 and r_4, are shown

Recursive force equations

The vectors $\mathbf{x}_\alpha$ and $\mathbf{x}_{\alpha+1}$ are related by the recursive form,

$$\mathbf{x}_{\alpha+1} = \mathbf{A}_\alpha \mathbf{x}_\alpha + \mathbf{B}_\alpha \mathbf{u}_\alpha, \qquad \alpha = 0, 1, 2, \cdots, q-1.$$

where

$$\mathbf{A}_\alpha \in \mathbb{R}^{2(\alpha+2)\times 2(\alpha+1)}, \quad \mathbf{B}_\alpha \in \mathbb{R}^{2(\alpha+2)\times(\alpha+2)}, \quad \mathbf{x}_\alpha \in \mathbb{R}^{2(\alpha+1)}, \quad \mathbf{u}_\alpha \in \mathbb{R}^{\alpha+2}.$$

It follows that

$$\mathbf{u}_\alpha = \begin{bmatrix} w_{\alpha+1,0} \\ w_{\alpha,1} \\ w_{\alpha-1,2} \\ w_{\alpha-2,3} \\ w_{\alpha-3,4} \\ \vdots \\ w_{0,\alpha+1} \end{bmatrix}, \qquad \mathbf{A}_\alpha = \begin{bmatrix} \mathbf{J}_2 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_1 \end{bmatrix},$$

$$\mathbf{B}_\alpha = \begin{bmatrix} \mathbf{\Phi}_{\alpha+1,0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{\Phi}_{\alpha,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{\Phi}_{\alpha-1,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{\Phi}_{\alpha-2,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{\Phi}_{0,\alpha+1} \end{bmatrix},$$

with

$$\mathbf{J} = \mathbf{\Omega} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \end{bmatrix}.$$

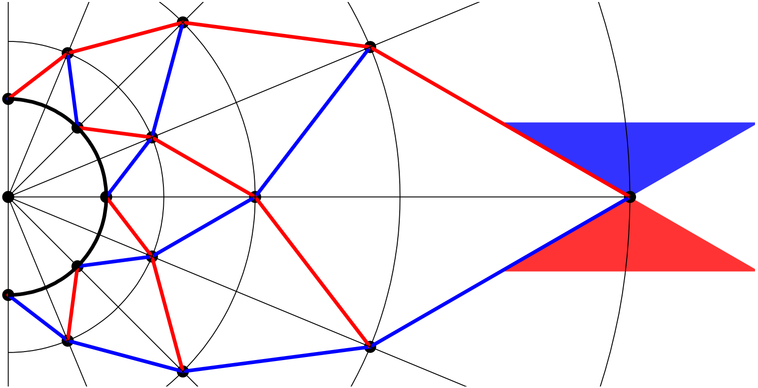
Note that all elements in $\mathbf{B}_\alpha$ have arguments in $\mathbf{\Phi}_{ik}$ such that $i+k=\alpha+1$, and all elements in $\mathbf{u}_\alpha$ have arguments w_{ik} such that $i+k=\alpha+1$.

Theorem for Equilibria of Michell Toplogies

Let a truss be arranged according to the Michell Topology of order q, having external forces $\mathbf{w}_{ik}$ applied at the nodes $\mathbf{n}_{ik}$ with $i \geq 0$, $j \geq 0$ and $i+j \leq q$. Let $\mathbf{x}_\alpha$ contain the forces (normalized by the member length) in all members within the band of members between radii r_α and $r_{\alpha+1}$ and $\mathbf{u}_\alpha$ contain the magnitude of the forces at the nodes with radius α. Then the forces propagate from one band to the next according to the linear recursive equation,

$$\mathbf{x}_{\alpha+1} = \mathbf{A}_\alpha \mathbf{x}_\alpha + \mathbf{B}_\alpha \mathbf{u}_\alpha, \qquad \alpha = 0, 1, 2, \cdots, q-1.$$

Loads at only one node



Michell Topology of Order 4 ($\phi = \pi/16$, $\beta = \pi/6$) showing bending region; blue and red indicate a member in compression or tension.
The members remain **uni-directionally** loaded for all forces within the region shown. These will be called **admissible forces**.

Material volume

If strings and bars are made from same material, then the total material volume of a Michell Topology of order q, with any **admissible force** is

$$J_q = (\bar{\lambda} + \bar{\gamma}) \sum_{i=0}^{q} \sum_{k=0}^{i} (t_{ik} - f_{ik}) p_{i+k}.$$

$$J_q = q\, r_0\, w_{00}\, (\bar{\lambda} + \bar{\gamma}) \frac{\sin|\theta|}{\sin\beta} \frac{\sin\phi}{\sin(\beta+\phi)},$$

where θ is the direction of the force, and w_{00} is the magnitude of the force. For any given material and external force, we need only minimize J'_q,

$$J'_q := \frac{J_q}{r_0\, w_{00}\, (\bar{\lambda} + \bar{\gamma}) \sin|\theta|} = \frac{q \sin\phi}{\sin\beta \sin(\beta+\phi)}$$

Define the aspect ratio ρ by

$$\rho := \frac{r_q}{r_0} = a^q = \left(\frac{\sin\beta}{\sin(\beta+\phi)}\right)^q,$$

Minimal Volume Solution

For any given aspect ratio ρ and any given complexity q, and any admissible force, J'_q is minimized by

$$\cos\phi^* = \left(\frac{2}{\rho^{-1/q}+\rho^{1/q}}\right), \qquad \tan\beta^* = \rho^{1/q}$$

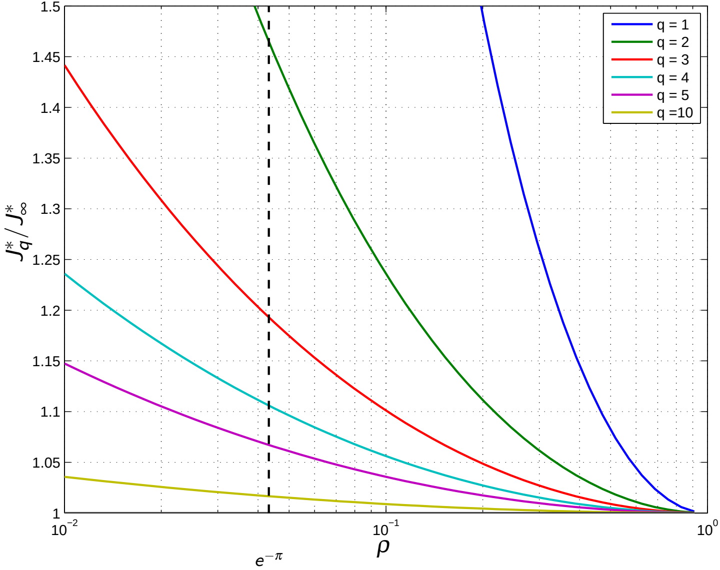
and the minimum volume of material required to build the structure is

$$J'^{\,*}_q = q\rho^{1/q}\sin\phi^*\left(1+\frac{1}{\tan^2\beta^*}\right) = q\,(\rho^{-1/q}-\rho^{1/q}).$$

As the chosen complexity q goes to ∞, $J'^{\,*}_\infty := \lim_{q\to\infty} J'^{\,*}_q$ can be computed as

$$J'^{\,*}_\infty = \lim_{q\to\infty} q\,(\rho^{-1/q}-\rho^{1/q}) = 2\ln\rho^{-1}.$$

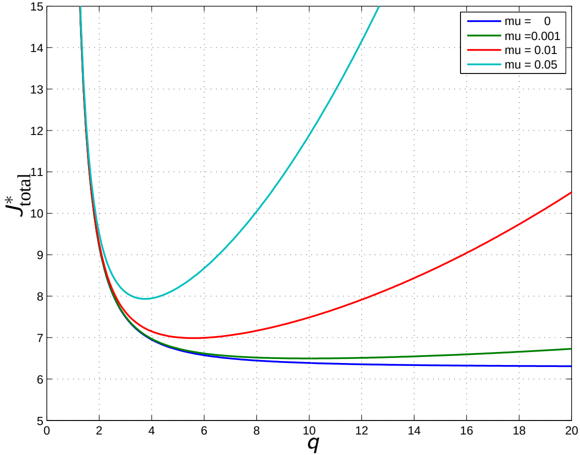
How does volume relate to complexity q and aspect ratio ρ?



The relative optimal discrete cost J_q^*/J_∞^*; material overlap occurs for $\rho \leq e^{-\pi}$

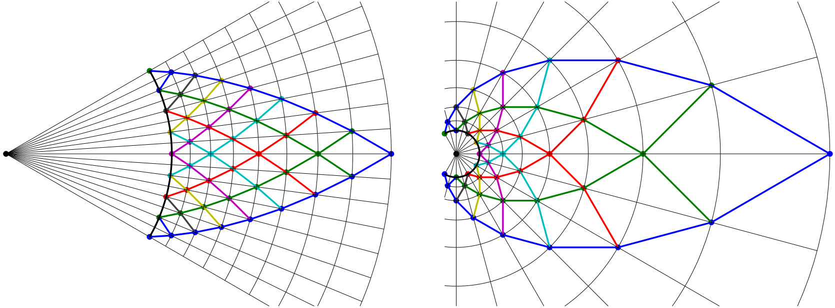
penalizing joint mass

$$J^*_{\text{total}} = {J'_q}^* + \mu\, q(q+1) = q\,(\rho^{1/q} - \rho^{-1/q}) + \mu\, q(q+1).$$



The joint mass optimal discrete cost J^*_{total} plotted for the worst case $\rho = e^{-\pi}$.

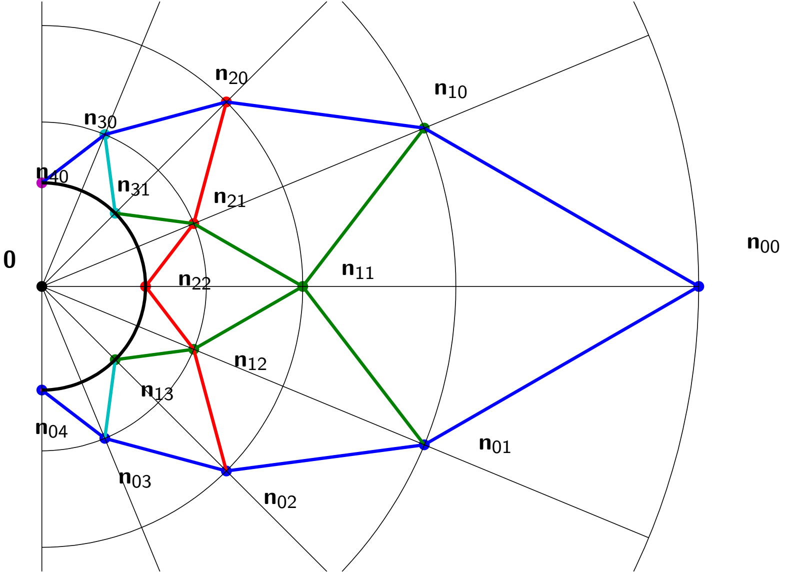
Optimal Michell Trusses of complexity $q = 8$



$(\rho = 0.59,\ q = 8),\ \Rightarrow$
$(\phi = 3.75^\circ,\ \beta = 43.125^\circ)$
$q\phi = 30^\circ$

$(\rho = 0.12,\ q = 8) \Rightarrow$
$(\phi = 15^\circ,\ \beta = 37.5^\circ)$
$q\phi = 300^\circ$

An Optimal Michell Truss of complexity $q = 4$



Optimal Michell Truss for ($q = 4$, $\rho = 0.199$), leads to ($\phi = 22.5^\circ$, $\beta = 33.75^\circ$)

Sub-Optimal Tensegrity bunk bed

Wall-mounted tensegrity bunk bed, with ($q = 5$, $\rho = 0.133$) leading to $\phi^* = 22.5^\circ$, $\beta^* = 33.5^\circ$ Sub-optimal due to truncated boundary (flat wall).

Optimal Compressive Structures

Hollow Cylinders in Compression

Buckling force $f(\ell_0)$ for a hollow tube with inner radius r_i and outer radius r_0, length ℓ_0, and mass $m(\ell_0)$

$$f(\ell_0) = \frac{E\pi^3(r_0^4 - r_i^4)}{4\ell_0^2}, \quad m(\ell_0) = \rho_b \pi (r_0^2 - r_i^2)\ell_0,$$

These equations yield the mass of the hollow cylinder $m(\ell_0)$

$$m(\ell_0) = \pi \rho_b \ell_0 r_i^2 \left(\sqrt{1 + \frac{\ell_0^2 f(\ell_0)}{E\pi^3 r_i^4}} - 1 \right)$$

where $m(\ell_0) \to 0$ as $r_i \to \infty$.
Therefore in FINITE environments a cylinder is not necessarily the minimal mass structure for compression!

Buckling Constraints for rods

Buckling force $f(\ell_0)$ for a rod of length ℓ_0, and mass $m(\ell_0)$, without external load

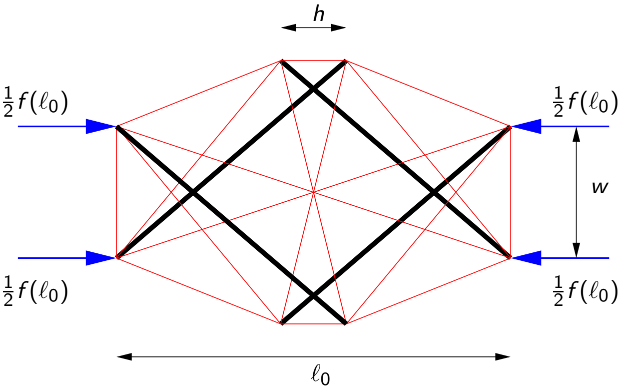
$$f(\ell_0) = \frac{E\pi^3 r_0^4}{4\ell_0^2}, \quad m(\ell_0) = \rho_b \pi r_0^2 \ell_0,$$

From these equations it follows that

$$m(\ell_0) = c_b \ell_0^2 \sqrt{f(\ell_0)}, \quad c_b = \frac{2\rho_b}{\sqrt{\pi E}}.$$

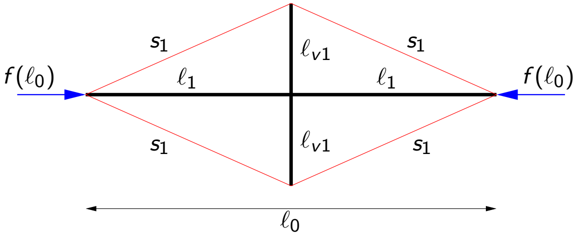
Existing Theory:
Fractal theory deals with the filling of space by replacing a given geometrical *object A* with yet another *object B* composed of some arrangements of object A of smaller dimensions.
New Theory:
Add a mechanical property requirement to fractals theory.
Assign a mechanical property to the geometrical object A (hence, *object A* will now be called *Structure A*) and replace *Structure A* by yet another *Structure B*, so that the specified mechanical property is preserved or improved.

A 4-bar, 16-string Class 1 Tensegrity

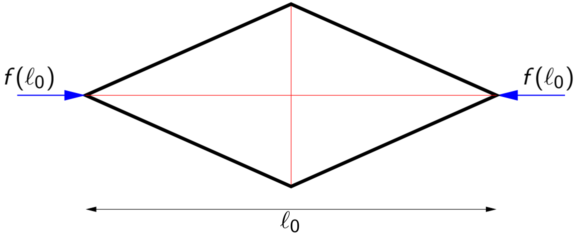


A 4-bar, 16 String structure under compressive load

A *T-Bar* Unit and its Dual, the *D-Bar* Unit

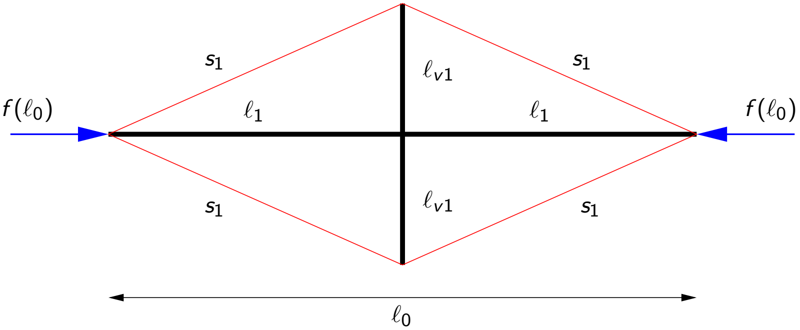


A *T-Bar* unit, a class-4 tensegrity



A *D-Bar* unit, (the *Dual* of the *T-Bar* unit) results when we choose $h = w = 0$. This is a class-2 tensegrity and there is a string-to-string connection.

A T-Bar structure under critical compressive load, $f(\ell_0)$.
Goal: Replace a bar of length ℓ_0 by a T-Bar tensegrity system.

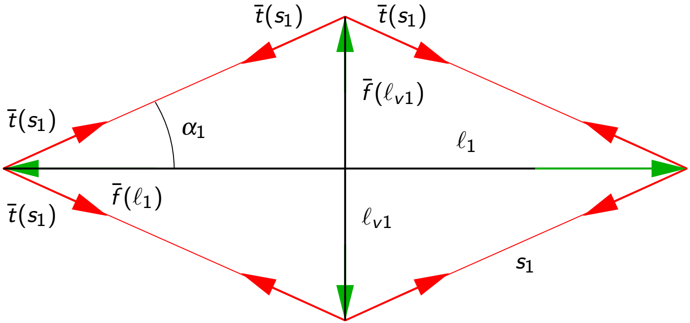


Assume a ball joint at the intersection of the 4 bars. (This is unstable in 3D, but stable in the 2D work here)

T-Bar Forces in the presence and absence of external loads

In the unloaded case ($f(\ell_0) = 0$), we require the same load $\bar{f}(\ell_1)$ in bar ℓ_1 as in the loaded case, $f(\ell_1)$ (where $f(\ell_0) \neq 0$, with $t(s_1) = 0$). To9 achieve this, for the unloaded (prestressed) case, $f(\ell_0) = 0$, and

$$\bar{f}(\ell_1) = f(\ell_0), \qquad \bar{f}(\ell_{v1}) = f(\ell_0)\tan\alpha_1, \qquad \bar{t}(s_1) = \frac{1}{2\cos\alpha_1} f(\ell_0).$$



For the loaded case,

$$f(\ell_{v1}) = 0, \qquad f(\ell_1) = f(\ell_0), \qquad t(s_1) = 0.$$

Mass of all bars in the *T-Bar* unit is m_{b_1}, where,

$$\begin{aligned} m_{b_1} &= 2m(\ell_{v1}) + 2m(\ell_1) \\ &= 2c_b(\ell_{v1})^2[(\bar{f}(\ell_{v1}))^{1/2} + (f(\ell_1))^{1/2}] \\ &= 2c_b(\ell_0/2)^2[\tan^2\alpha_1(f(\ell_0)\tan\alpha_1)^{1/2} + (f(\ell_0))^{1/2}] \\ &= 2c_b\,(\ell_0/2)^2\,(f(\ell_0))^{1/2}[\tan^{5/2}\alpha_1 + 1]. \end{aligned}$$

Hence, the mass of the *T-Bar* unit is less than the mass of the original bar if $\mu_{b_1} < 1$, where,

$$\mu_{b1} = \frac{m_{b_1}}{m(\ell_0)} = \frac{1}{2}[\tan^{5/2}\alpha_1 + 1],$$

where it is clear that $\mu_{b_1} < 1$ if $\alpha_1 < 45^\circ$.

The string mass is $4m(s_1) = 4c_s s_1 \bar{t}(s_1)$, and $\bar{t}(s_1) = f(\ell_0)/(2\cos\alpha_1)$ is the string tension in the externally unforced case. The total mass ratio is

$$
\begin{aligned}
\mu_1 &= \frac{1}{2}(\tan^{5/2}\alpha_1 + 1) + 4m(s_1)/m(\ell_0) \\
&= \frac{1}{2}(\tan^{5/2}\alpha_1 + 1) + 4\frac{c_s}{m(\ell_0)}\left(\frac{\ell_0}{2\cos\alpha_1}\right)\left(\frac{f(\ell_0)}{2\cos\alpha_1}\right) \\
&= \frac{1}{2}(\tan^{5/2}\alpha_1 + 1) + 4\left(\frac{c_s\sqrt{f(\ell_0)}}{c_b\ell_0}\right)\left(\frac{1}{2\cos\alpha_1}\right)^2 \\
&= \frac{1}{2}(\tan^{5/2}\alpha_1 + 1) + \varepsilon(1 + \tan^2\alpha_1),
\end{aligned}
$$

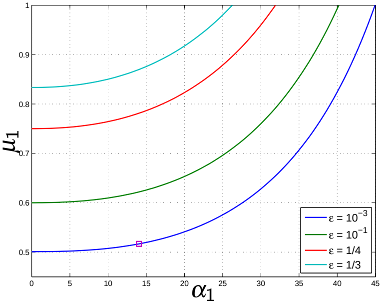
where

$$
\varepsilon = \frac{c_s\sqrt{f(\ell_0)}}{c_b\ell_0} = \frac{\rho_s\sqrt{\pi E f(\ell_0)}}{2\sigma_s\rho_b\ell_0}
$$

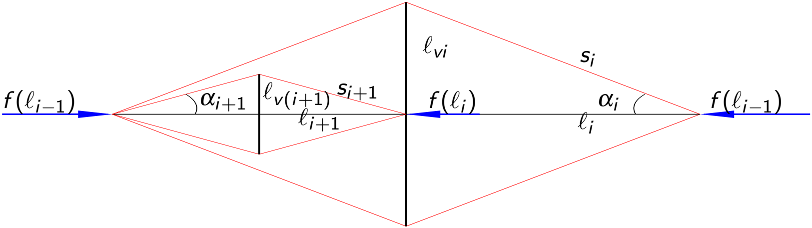
is a dimensional parameter.

Since $\mu_1 \geq 1/2+\varepsilon$, for $\mu_1 \leq 1$, $\varepsilon \leq 1/2$ is required, yielding:

$$\frac{f(\ell_0)}{\ell_0^2} < \left(\frac{\rho_b^2}{\pi E}\right)\left(\frac{\sigma_s}{\rho_s}\right)^2.$$



For steel $c_s/c_b = 0.5829\times 10^{-3}$, $f(\ell_0) = 2.942, \ell_0^2$ Newtons, we have $\varepsilon = 0.001$, choosing $\tan\alpha_1 = 0.25$, then $\mu_1 = 0.517$. This point is marked with a square in the plot.



$$\ell_{i+1} = \frac{1}{2}\ell_i, \qquad s_i = \ell_i / \cos\alpha_i, \qquad \ell_{vi} = \ell_i \tan\alpha_i,$$

$$f(\ell_i) = f(\ell_{i-1}) + 2t(s_i)\cos\alpha_i, \qquad f(\ell_{vi}) = 2t(s_i)\sin\alpha_i,$$

$$\frac{m(\ell_i)}{m(\ell_0)} = \left(\frac{\ell_i}{\ell_0}\right)^2 \sqrt{\frac{f(\ell_i)}{f(\ell_0)}}.$$

Choosing $t(s_1) = 0$, yields $f(\ell_i) = f(\ell_{i-1})$, and $f(\ell_{vi}) = 0$.

Mass of Bars ℓ_n After n Self-Similar Iterations

After n iterations the number of bars of length ℓ_n is 2^n,
The number of bars of lengths $\ell_{v1}, \ell_{v2}, \ldots \ell_{vn}$ sum to $\sum_{i=1}^{n} 2^i$ bars.
After n iterations, the total mass of the bars ℓ_n is given by,

$$2^n m(\ell_n) = c_b 2^n (\ell_n)^2 \sqrt{f(\ell_n)} = c_b 2^n \left(\frac{\ell_0}{2^n}\right)^2 \sqrt{f(\ell_0)}.$$

Hence,

$$\frac{2^n m(\ell_n)}{m(\ell_0)} = \frac{1}{2^n}.$$

Total Mass After n Iterations

Mass of bars ℓ_{vi}

$$\sum_{i=1}^{n} 2^i m(\ell_{vi})/m(\ell_0) = \sum_{i=1}^{n} 2^i \left(\frac{\tan^{5/2}\alpha_i}{2^{2i}} \right) = \sum_{i=1}^{n} \frac{\tan^{5/2}\alpha_i}{2^i}$$

Mass of the strings s_i.

$$\sum_{i=1}^{n} 2^{i+1} m(s_i)/m(\ell_0) = \sum_{i=1}^{n} \varepsilon(1+\tan^2\alpha_i),$$

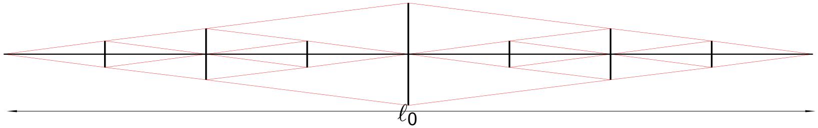
Total mass after n iterations:

$$\mu_n = m_n/m(\ell_0) = \frac{1}{2^n} + \sum_{i=1}^{n} \frac{\tan^{5/2}\alpha_i}{2^i} + \sum_{i=1}^{n} \varepsilon(1+\tan^2\alpha_i).$$

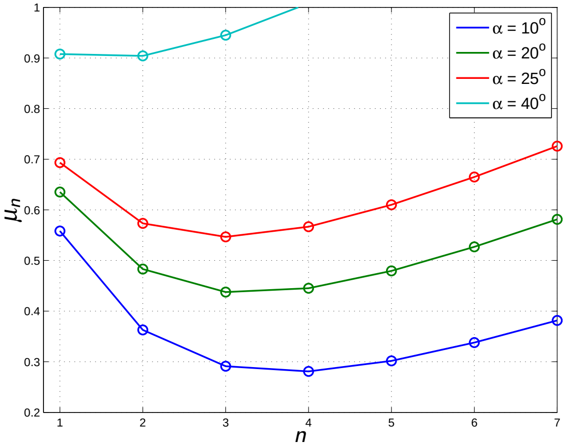
Special case: If we choose $\alpha_i = \alpha$, then the total mass ratio is

$$\mu_n(\alpha_i = \alpha) = 2^{-n} + \tan^{5/2}\alpha(1-2^{-n}) + n\varepsilon(1+\tan^2\alpha).$$

Constant α, for $n = 4$



Mass vs Self-Similar Iterations, with Constant α



$\varepsilon = 0.05$.

As $n \to \infty$, tendon mass $\to \infty$ and bar mass ratio $\to (\tan^{5/2}\alpha)$.

Optimal Complexity

Pretend that n is any real number, rather than an integer. Compute $\frac{\partial \mu_n}{\partial n} = 0$ to obtain:

$$2^{n^*} = \frac{\ln 2(1 - \tan^{5/2}\alpha)}{(1 + \tan^2\alpha)\varepsilon},$$

or, explicitly, rounding down to the nearest integer,

$$n^* = \lfloor \frac{\ln[(\ln 2)(1 - \tan^{5/2}\alpha)] - \ln[(1 + \tan^2\alpha)\varepsilon]}{\ln 2} \rfloor,$$

The number of tensegrity components required to build the minimal mass structure in compression is

$$\begin{aligned} q &= 2^{n^*} + \sum_{i=1}^{n^*}(2^i + 2^{i+1}) \\ &= 3(2^{n^*}) - 2. \end{aligned}$$

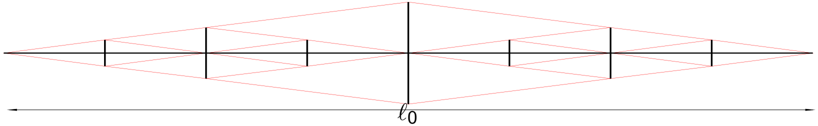
Example

For steel strings and bars $c_s/c_b = 0.5829 \times 10^{-3}$. With external force $f(\ell_0) = 2.942\ell_0^2$ Newtons, we have $\varepsilon = 0.001$, and from the above equation, choosing $n = 6$ and $\tan\alpha = 0.25$, we obtain $\mu_6 = 0.0528$, indicating a structure of $q = 190$ components that has about $1/20$th the mass the original bar of length ℓ_0, and the same buckling strength.

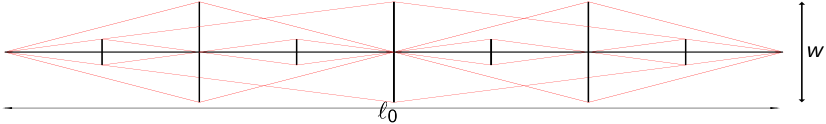
Example

Verify that the optimal mass in the above example occurs with a structure of complexity $n^* = 9$, in which case the minimal mass is 0.0427 times the mass of the original bar.

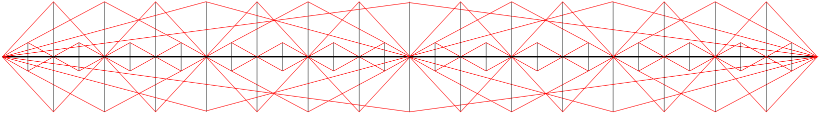
Constant Width Column



T-Bar self-similar $n = 4$, constant α



Constant-Width T-Bar column $\ell_0/w = 2.3$, $n = 3$, $k = 2$, $[\alpha_i \leq 45^o$, $i = 1, \ldots, k]$, $[\alpha_i = \alpha_1$, $i = k+1, \ldots, n]$



Optimal constant width T-Bar column $\ell_0/w = 10$, $n^* = 5$, $k = 4$

Yielding in T-Bar Self-Similar Systems

The bar ℓ_n yields when the bar force is $f(\ell_n) = \sigma_b \pi r_n^2$, or, equivalently

$$r_n^2 = f(\ell_n)/(\pi\sigma_b) = f(\ell_0)/(\pi\sigma_b).$$

The bar buckles when the bar force is $f(\ell_n) = f(\ell_0) = \pi^3 E r_n^4/(4\ell_n^2)$, or equivalently, (using the fact $\ell_n = \ell_0/(2^n)$),

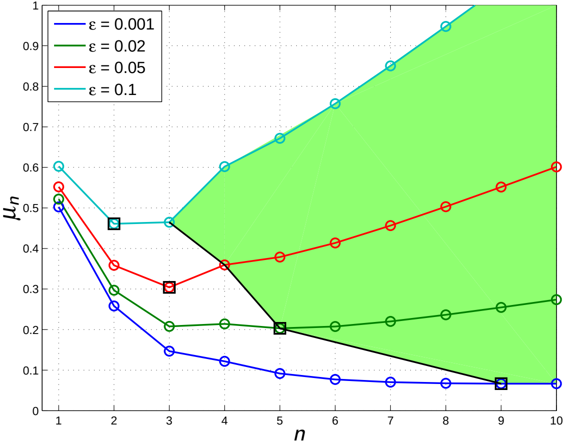
$$r_n^2 = 2^{n+1}\ell_0\sqrt{f(\ell_0)/(\pi^3 E)}.$$

Equating these expressions for r_n yields the iteration number n^{**} at which the buckling force and the yield force are the same, for bar ℓ_n.

$$2^{n^{**}} = \frac{2\ell_0\sigma_b}{\sqrt{\pi E f(\ell_0)}} = \frac{(\rho_s/\sigma_s)}{(\rho_b/\sigma_b)}\frac{1}{\varepsilon}.$$

If $n > n^{**}$ the failure of bar ℓ_n is by yielding.
If $n < n^{**}$, the failure of bar ℓ_n is by buckling, in which case

$$2^{n^*} = \frac{\ln 2(1-\tan^{5/2}\alpha)}{(1+\tan^2\alpha)\varepsilon},$$

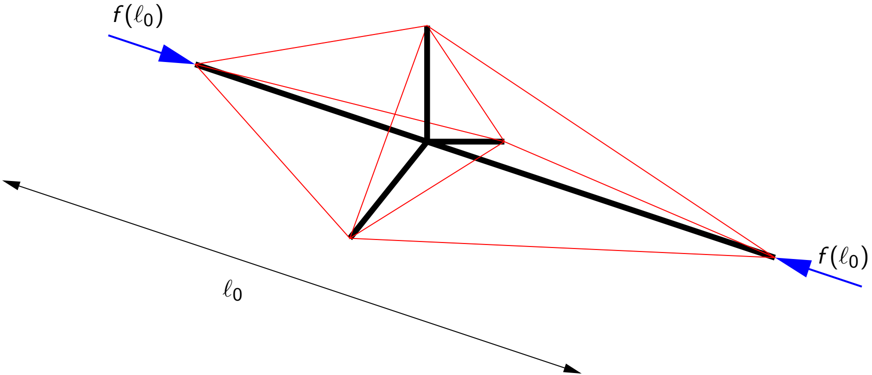


$\ell_0/w = 10$; yielding is the mode of failure in green area.

Example

For *T-Bar* columns with steel materials:
The number of self-similar iterations before yielding satisfies
$2^{n^{**}} = 232.9(f(\ell_0)^{-1/4}$,
where $f(\ell_0)$ is the external force applied.
Hence, for steel materials with $f(\ell_0) = 1$, the largest number of self-similar iterations before yielding is 7.

3D T-Bar Systems, with N-Polygonal cross-sections



A T-Bar Unit with N=3

For any N, the mass is

$$\mu_n^N = \frac{1}{2^n} + \frac{N}{2}\sum_{i=1}^{n} \frac{\tan^{5/2}\alpha_i}{2^i} + \frac{N}{2}\sum_{i=1}^{n} \varepsilon(1+\tan^2\alpha_i).$$

where, for constant α

$$\mu_n^N(\alpha_i = \alpha) = 2^{-n} + (N/2)\tan^{5/2}\alpha(1-2^{-n}) + (N/2)n\varepsilon(1+\tan^2\alpha).$$

Differentiating this with respect to n and solving for n^* yields the optimal complexity

$$2^{n^*} = \frac{2\ln 2(1-\frac{N}{2}\tan^{5/2}\alpha)}{\varepsilon N(1+\tan^2\alpha)}.$$

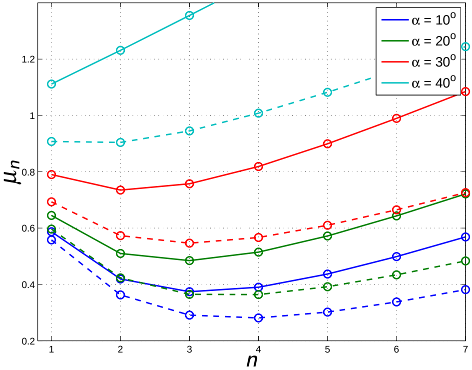
Now substitute 2^{n^*} into $\mu_n^N(\alpha_i = \alpha)$ to get the minimal mass

$$\mu_{n^*}^N = \frac{N}{2}\tan^{5/2} + \varepsilon \frac{N(1+\tan^2\alpha)}{2\ln 2}\left[1 + \ln\left(\frac{2\ln 2(1-\frac{N}{2}\tan^{5/2}\alpha)}{\varepsilon N(1+\tan^2\alpha)}\right)\right].$$

Observe that the total bar mass is smallest for $N = 3$.
For $N = 3$, constant $\alpha_i = \alpha$, and any n:

$$\mu_n^3(\alpha_i = \alpha) = 2^{-n} + (3/2)\tan^{5/2}\alpha(1-2^{-n}) + (3/2)n\varepsilon(1+\tan^2\alpha).$$

Mass of T-Bar Systems for N=2 and N=3

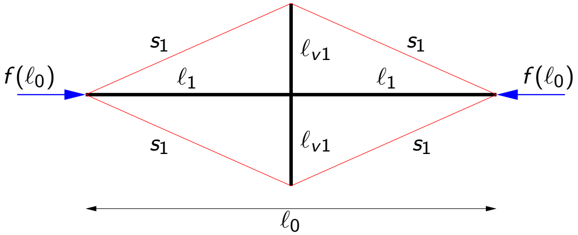


$\varepsilon = 0.05$.

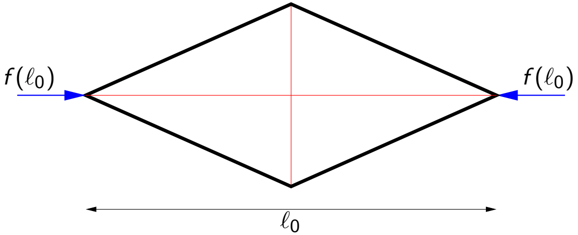
$(N = 2$, dashed) and $(N = 3$, solid), The plot for $N = 3$ has solid lines. Dashed lines for $N = 2$

Note for $N = 3$ that α must be at least 4.63° smaller than for the $N = 2$ case (to get $\mu_n^N \leq 1$ when $\varepsilon = 0$), requiring $\tan^{5/2}\alpha \leq 2/N$.

The Dual of the A *T-Bar* Unit: the *D-Bar* Unit

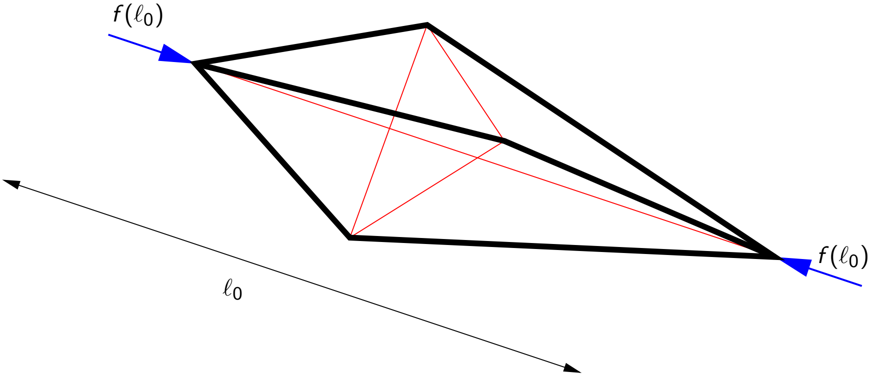


A *T-Bar* unit, a class-4 tensegrity

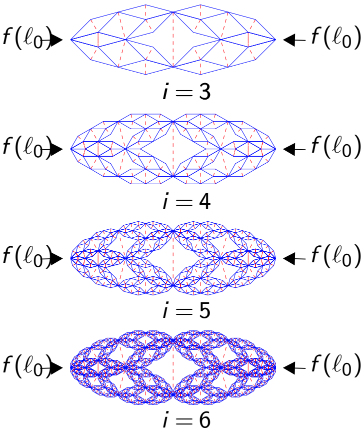


A *D-Bar* unit, (the *Dual* of the *T-Bar* unit) results when we choose $h = w = 0$. This is a class-2 tensegrity and there is a string-to-string connection.

The 3D D-Bar Unit



Self-Similar D-Bar Systems



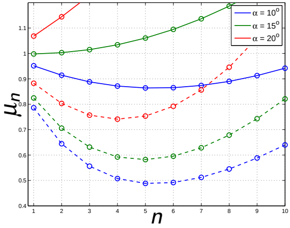
Configurations of the planar *D-Bar* Self-Similar structure with constant $\alpha = 15^\circ$. The s_{vi} strings are red. The s_i strings are not shown since they take no tension in these critical states

$$\mu_n^N = \left(\frac{N}{4\cos^5\alpha}\right)^{n/2} + \varepsilon\,(\cos^{-2n}\alpha - 1)/\sin^2\alpha.$$

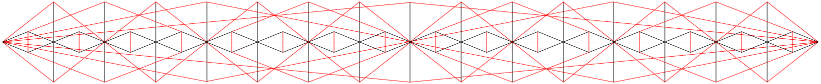
The string mass does not depend on N. For $n=1$, $N=3$:

$$\mu_1^3 = \left(\frac{3}{4\cos^5\alpha}\right)^{1/2} + \varepsilon\,(1+\tan^2\alpha).$$

$\mu_1^3 < 1$ requires $\alpha < 19.25^\circ$ for N=3, and $\alpha < 29.48^\circ$ for N=2. The plot compares ($N=2$, dashed) and ($N=3$, solid), with $\varepsilon = 0.05$.



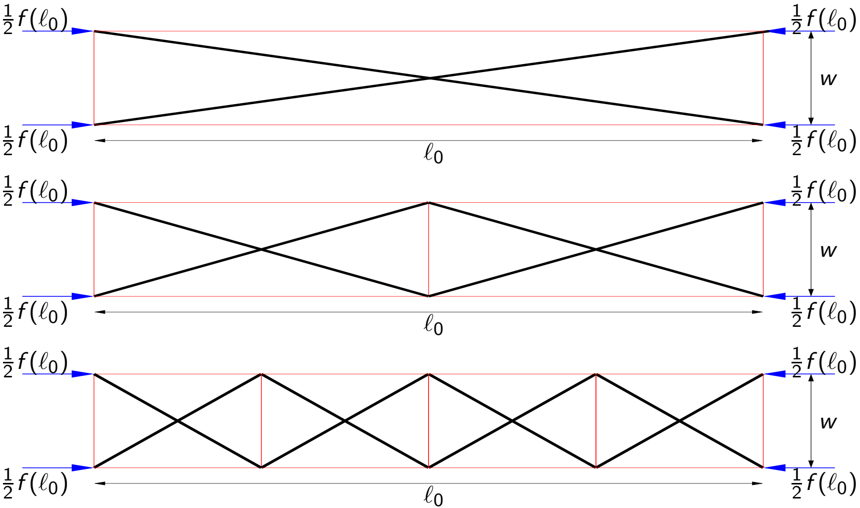
Optimal constant width T-Bar column $\ell_0/w = 10$, $n^* = 5$, $k = 4$, with last iteration using D-Bar units



It is easy to collapse a *D-Bar* unit by controlling the string s_{v1}, while the collapse of the *T-Bar* unit requires a folding procedure which seems more complex. Yet the *T-Bar* self-similar system can reduce more mass on each iteration than the *D-Bar* system. To combine the advantages of both, one can use the *T-Bar* self-similar iteration except on the last iteration, where *D-Bar* units will be employed.

PREVIOUS: REPLACING EACH BAR ELEMENT WITH A SELF-SIMILAR UNIT
AHEAD: REPLACING EACH UNIT WITH TWO SIMILAR UNITS

Unit-Self-Similar Designs with Box Units



Under these compressive loads, how many Unit-Self-similar iterations give minimal mass?

The sum of forces at the typical node yields,

$$f(\ell_n)\sin\alpha_n = t(s), \qquad\qquad f(\ell_0) = f(\ell_n)\cos\alpha_n.$$

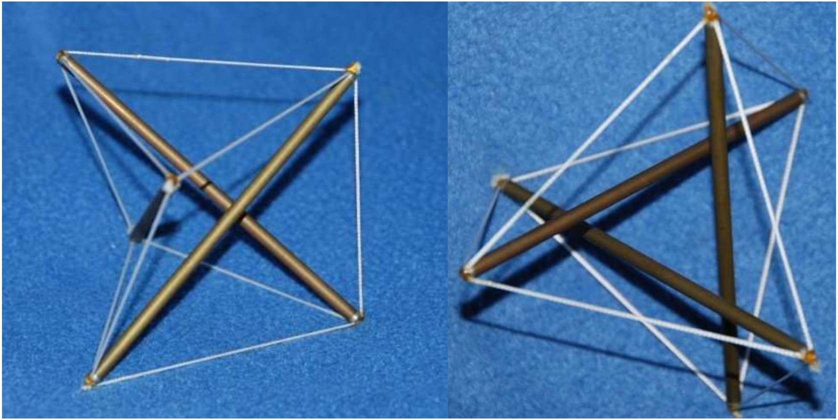
and the mass is composed of $2n$ bars, leading to

$$\begin{aligned} m &= 2nc_b\ell_n^2\sqrt{f(\ell_n)} \\ &= 2nc_b[w^2+(\ell_0/n)^2]\{f(\ell_0)^2[1+(nw/\ell_0)^2]\}^{1/4}. \end{aligned}$$

Differentiating this mass expression with respect to n yields the optimal complexity,

$$n = (\ell_0/w)\sqrt{2/3}.$$

Regular Tensegrity Prisms



A regular minimal 3-bar tensegrity prism
regular = tops and bottoms have same vertical centerline and are parallel.
minimal = stabilized with smallest number of strings possible

Equilibrium for a Regular Tensegrity Prism

$\gamma_t,\ \gamma_b,\ \gamma_v :=$ force densities in strings $s_t,\ s_b,\ s_v$.
$\lambda_b :=$ force density in each bar

$$\begin{pmatrix}\gamma_t\\ \gamma_b\\ \gamma_v\end{pmatrix} = \lambda_b \begin{pmatrix}\rho^{-1}(2\sin(\pi/p))^{-1}\\ \rho\,(2\sin(\pi/p))^{-1}\\ 1\end{pmatrix},$$

where $\rho := r_t/r_b$ is the ratio of top and bottom radii.
The twist angle is

$$\alpha = \frac{\pi}{2} - \frac{\pi}{p},$$

for $p = 3$ then $\alpha = 30^\circ$,
for $p = 4$ then $\alpha = 45^\circ$,
for $p = 6$ then $\alpha = 60^\circ$
Also note that $\gamma_b = \rho^2\gamma_t$.

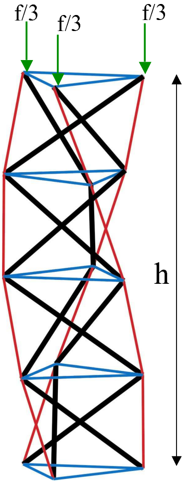
Equilibria and Mass for a Single Unit

Force densities in all members,

$$\lambda_b = \gamma_v = \frac{f(\ell_0)}{\ell_0\, p}, \qquad \gamma_t = \gamma_b = \frac{\lambda_b}{2\sin(\pi/p)} = \frac{f(\ell_0)}{2\ \ell_0\, p \sin(\pi/p)},$$

Total normalized mass for a single minimal regular tensegrity p-bar prism:

$$\mu_1 = \sqrt{p}\left(\frac{1+2z^4+\sin(\pi/p)}{2}\right)^{5/4} + \varepsilon\left(1+\frac{1+\sin(\pi/p)}{2z^4}\right).$$



Perspective view:

$r_t = r_b = r$ and stable minimal regular p-bar prisms can be stacked as shown.

bar and string mass:

$$\mu_n = \mu_{bn} + \mu_{sn},$$

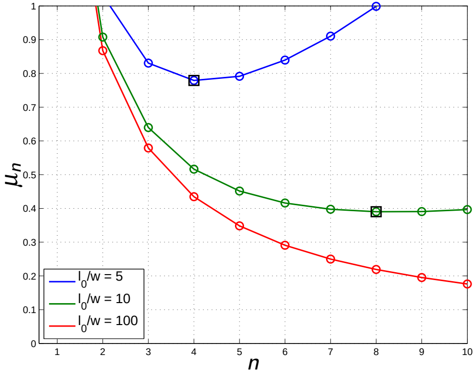
$$\mu_{bn} = \frac{\sqrt{p}}{nz^5}\left(\frac{n^2 + 2z^4 + n^2\sin(\pi/p)}{2}\right)^{5/4},$$

$$\mu_{sn} = \varepsilon\left(1 + \frac{n^2[1+\sin(\pi/p)]}{2z^4}\right).$$

differentiating μ_{bn} with respect to n yields optimal bar complexity:

$$n^* = \left\lfloor \frac{2z^2}{\sqrt{3[1+\sin(\pi/p)]}} \right\rfloor = \left\lfloor \frac{2}{\sqrt{3(1+\sin(\pi/p)}}\frac{\ell_0}{w} \right\rfloor.$$

Example: Unit-self-similar 3-bar prism, $\varepsilon = 0.001$



Note from properties of the single prism, one needs $n \geq 2$ to reduce mass. The points marked with a square indicate global minima. The global minimum for $\ell_0/w = 100$ is $\mu_n = 0.0405$ at $n = 82$.

Note that

$$n^*(\ell_0/w) = \{4, 8, 84\}, \qquad \ell_0/w = \{5, 10, 100\}$$

indicating that the optimal bar complexity n^* is a reasonable estimate of the global optimum (hence, bars dominate the mass).

Substitute the optimal bar complexity, n^*, into the mass formula to get:

$$\mu_{n^*} = \frac{1}{\ell_0/w}\frac{5\times 15^{1/4}}{6}\sqrt{p[1+\sin(\pi/p)]}+\varepsilon\frac{10}{6}.$$

This remarkable formula shows that the optimal mass is linear in ε and that the string mass is independent of p, the number of bars per prism . Also, for a given p, the optimal mass gain is inversely proportional to the aspect ratio ℓ_0/w.
Example: Show that for $p=3$ and $p=6$, that mass cannot be reduced unless $\ell_0/w>4$ and $\ell_0/w>5$, respectively.

Mass of Minimal Regular Tensegrity plates

$$\mu_n = \mu_{bn} + \mu_{sn} \qquad \mu_{bn} = \frac{\sqrt{3}}{n^{3/4}\pi^{5/4}} \left(n\pi + \xi^{-1}(2+\sqrt{3})z^4\right)^{5/4},$$

$$\mu_{sn} = \varepsilon \left(1 + \frac{2+\sqrt{3}}{n\pi\xi} z^4\right).$$

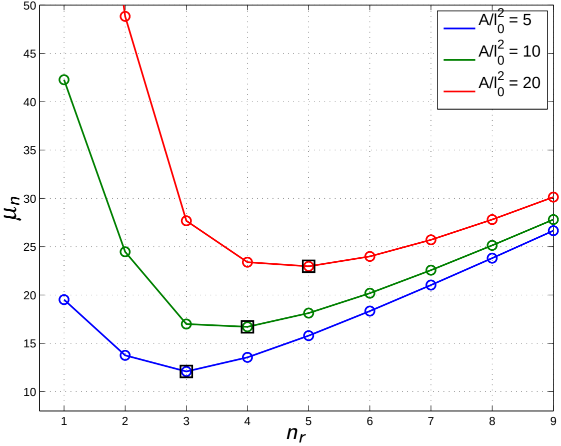
$$\xi(n_r) = \frac{A}{na} = \frac{R^2}{nr^2} = \frac{[1+2\|t_3\|(n_r-1)]^2}{4[1+3(n_r-1)n_r]}.$$

For example, $\xi(n_r)$ is approximately equal to

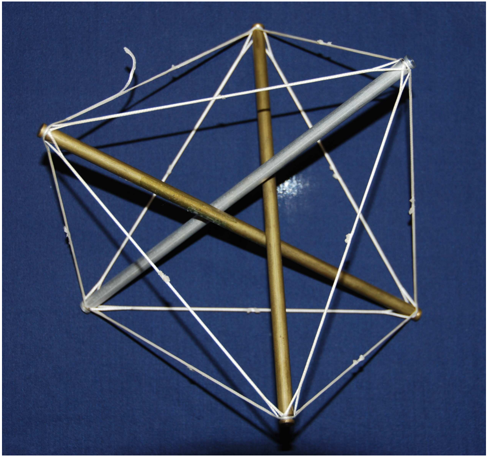
$$\{0.60, \quad 0.68, \quad 0.75, \quad 0.80\}$$

when $n_r \to \{2, 3, 6, \infty\}$

3-Bar Plate Mass versus complexity



$\varepsilon = 0.05$ and $A/\ell_0^2 = \{5, 10, 20\}$.



A regular non-minimal tensegrity prism. This structure has 12 strings, including 3 extra diagonal strings

Let γ_d be the force density in the diagonal strings then

$$\begin{pmatrix} \gamma_t \\ \gamma_b \\ \gamma_v \\ \gamma_d \end{pmatrix} = \frac{\lambda_b}{\cos(\alpha - \pi/p)} \begin{pmatrix} \rho^{-1}\cos(\pi/p) \\ \rho\cos(\pi/p) \\ 2\cos(\alpha)\cos(\pi/p) \\ -\cos(\alpha + \pi/p) \end{pmatrix}$$

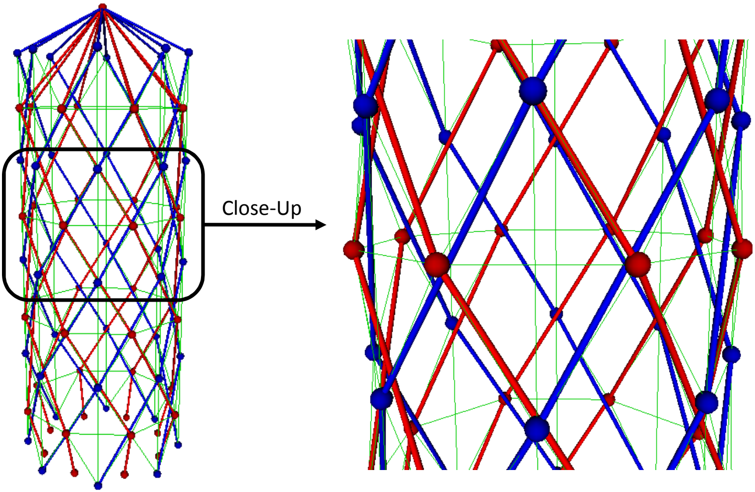
where now the twist angle is any angle between

$$\frac{\pi}{2} - \frac{\pi}{p} \leq \alpha \leq \frac{\pi}{2}.$$

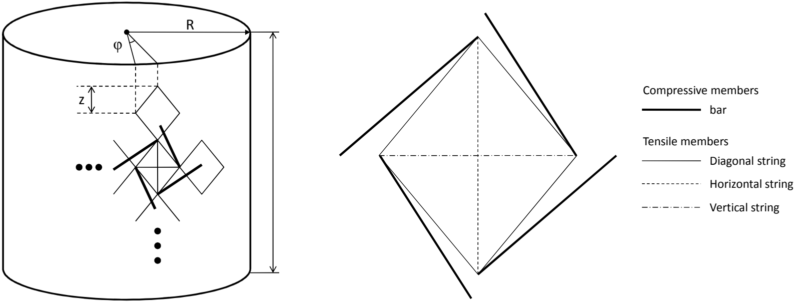
In this range, the γ's are all non-negative for $\lambda_b > 0$. As in the minimal regular prism, $\gamma_b/\gamma_t = \rho^2$.
Note also that $\alpha = \pi/2 - \pi/p$ is the equilibrium for the Minumal Regular Prism. The above result yields $\gamma_d = 0$ when $\alpha = \pi/2 - \pi/p$.
It is important to note that all tensegrity columns and plates in the previous discussions can be designed using Non-minimal regular Tensegrity Prisms. The difference is advantage is high stiffness, with *NO soft modes.*

Minimal Mass Cylinder

- This section provides an explicit analytical solution to the minimal mass design for a special class of long cylindrical structures.
- The mass is minimized subject to material yield constraints for an infinitely long cylinder composed of axially loaded compressive members and tensile members.
- The topology of the structure is a class 2 tensegrity structure, meaning that two compressive members are in contact at each node.



Class 2 tensegrity cylinder



Overall View Typical Unit

Diagram of class 2 tensegrity cylinder

Define:
$p\phi = 2\pi$, where $\phi =$ angle between radial lines
$z =$ height of each stage
$H = qz =$ height of cylinder
$R =$ radius of the cylinder
$l_b =$ length of each bar
$l_s =$ length of each of the diagonal strings
$l_h =$ length of the horizontal string in each unit
$l_v =$ length of the vertical string in each unit.
Then,

$$\begin{aligned} l_b^2 &= 4R^2 \sin^2\phi + 4z^2 \\ l_s^2 &= 2R^2(1-\cos\phi) + z^2 \\ l_h^2 &= 4R^2 \sin^2\phi \\ l_v^2 &= 4z^2, \end{aligned}$$

Main Result: Equilibria with vertical external loads

All bars have equal force in the infinite cylinder,
Summing forces at only two nodes yields the fact:
The force densities in the horizontal string, γ_h, and in the diagonal strings γ_1, γ_2, γ_3, γ_4 may be assigned any positive values, subject to the constraint,

$$\gamma_1 = \gamma_3, \qquad \gamma_4 = \gamma_2,$$

The force density λ in each bar is given by

$$\lambda = \gamma_h + \frac{1}{2(1+\cos\phi)}(\gamma_1 + \gamma_4).$$

Minimizing Structural Mass

A well-known fact:
Minimal mass, subject to material yield constraints in all bars and strings, is achieved by minimizing the function

$$J = \sum_{i=1}^{m} \sigma_i l_i^2,$$

where σ is the force density in the ith member (bar or string), and l_i is the length of the ith member.

Using the above results,

$$J = J_s + J_b,$$

where J_s and J_b are the normalized mass of strings and bars, respectively. The total mass of strings and bars are given by:

$$J_s = c_s \left[\frac{1}{2} pq(\gamma_1 + \gamma_4) l_s^2 + \frac{1}{2}(q+1) p \gamma_h l_h^2 + \frac{1}{2} pq \gamma_v l_v^2 \right]$$

$$J_b = c_b \left[\frac{1}{2} pq \lambda l_b^2 \right].$$

Optimal Complexity

The partial differential of J with respect to complexity q can be obtained as follows:

$$\frac{\partial J}{\partial q} = \frac{\partial J_s}{\partial q} + \frac{\partial J_b}{\partial q}$$

where,

$$\begin{aligned} \frac{\partial J_s}{\partial q} &= c_s \left[\frac{1}{2}(\gamma_1 + \gamma_4) p (2R^2(1-\cos\phi) - z^2) + 2R^2 \gamma_h p \sin^2\phi - 2\gamma_v p z^2 \right] \\ \frac{\partial J_b}{\partial q} &= c_b \left[2p\lambda(R^2 \sin^2\phi - z^2) \right] \end{aligned}$$

From $\partial J / \partial q = 0$, it follows that

$$q^* = \sqrt{2} \left(\frac{H}{2R} \right) \sqrt{\frac{4(r_c r_{\gamma_v} + r_{\gamma_h}) + r_c + 2(1+\cos\phi)^{-1}}{(r_c+1)(2r_{\gamma_h}\sin^2\phi - \cos\phi + 1)}}$$

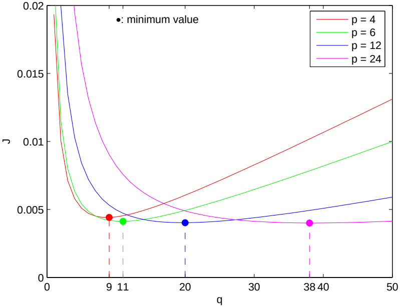
where, q^* is the optimal value of q, which minimizes the mass of structure.

Optimal System with Same Material for Strings and Bars

For same bar and string material, and:
$r_c = 1$
only diagonal strings are present, then $r_{\gamma_h} = r_{\gamma_v} = 0$ and,

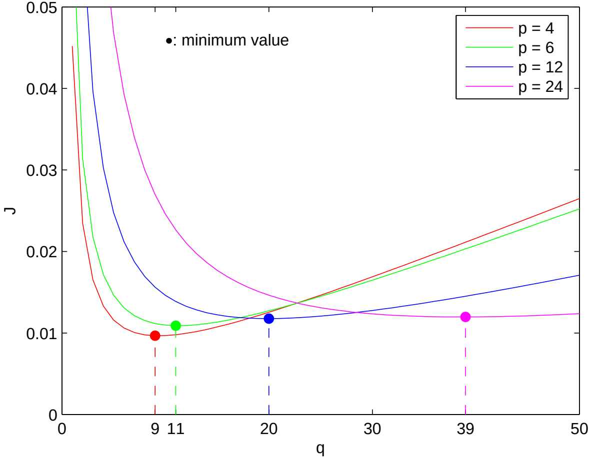
$$q^* = \left(\frac{H}{2R}\right)\sqrt{\frac{3+\cos\phi}{\sin^2\phi}}$$

Cylinder Mass versus Complexity



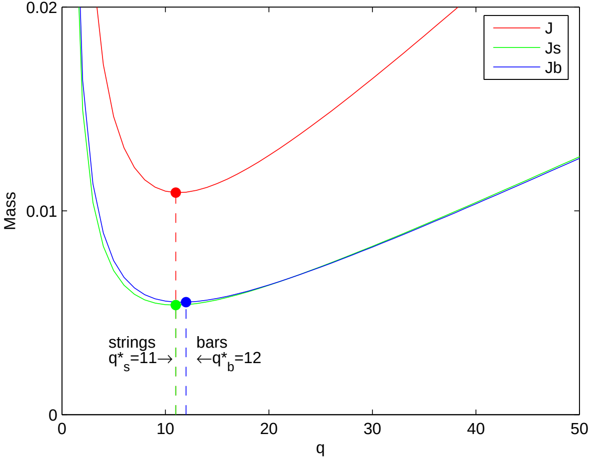
J versus q for the case ($r_{\gamma_h} = r_{\gamma_v} = 0,\ r_c = 1,\ \frac{H}{2R} = 5,\ R = 0.1$)

Cylinder Mass versus Complexity

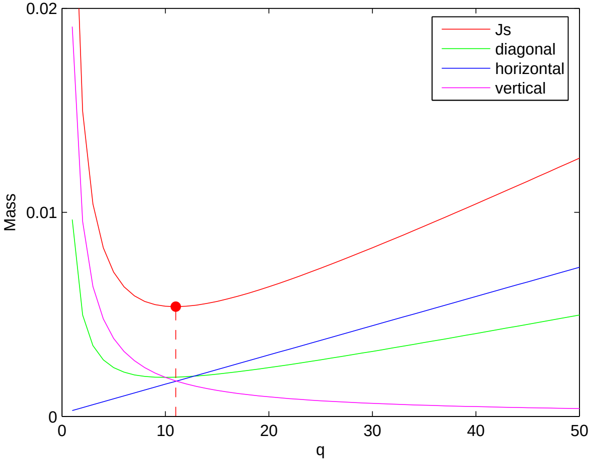


J versus q for the case $\left(r_{\gamma_h} = r_{\gamma_v} = 0.5,\ r_c = 1,\ \frac{H}{2R} = 5,\ R = 0.1\right)$

Bar Mass versus String Mass

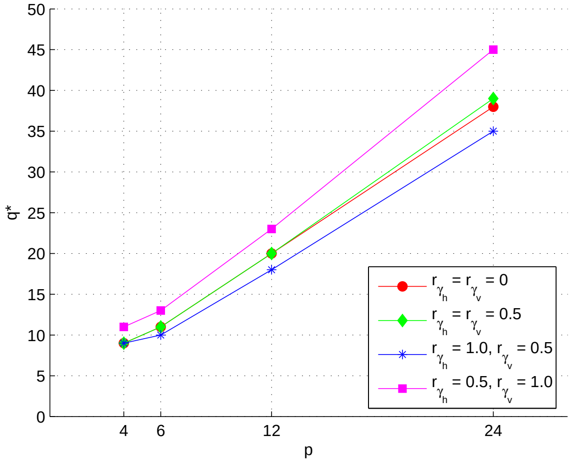


Ratio of mass of bars and strings for the case:
$(p = 6,\ r_{\gamma_h} = r_{\gamma_v} = 0.5,\ r_c = 1,\ \frac{H}{2R} = 5,\ R = 0.1)$



Mass of diagonal, horizontal and vertical strings for the case: $(p = 6,\ r_{\gamma_h} = r_{\gamma_v} = 0.5,\ r_c = 1,\ \frac{H}{2R} = 5,\ R = 0.1)$

Optimal Complexity



$p - q^*$ $(r_c = 1, \frac{H}{2R} = 5)$

Efficient Models of Class 1 Tensegrity dynamics

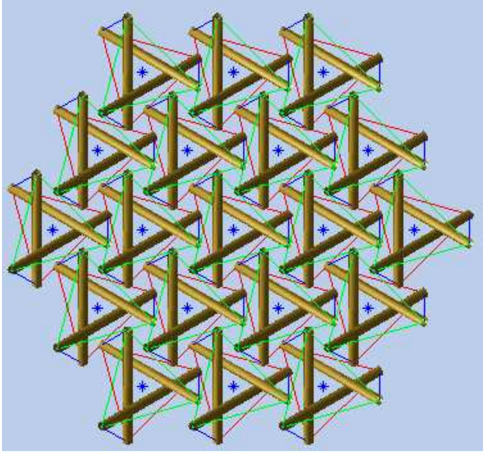
- A tensegrity system requires prestress for stabilization of the configuration of rigid bodies and tensile members.
- Provide an efficient model for both static and dynamic behavior of such systems, specialized for the case when the rigid bodies are axi-symmetric rods.
- The key to efficient nonlinear dynamic models, and
- The key to effective feedback control of the nonlinear system, is to:
- Find and *exploit* the special structure of the model equations.

Vector and Matrix forms

$$M(q)\ddot{q} + G(q)\dot{q} + K(q)q = Hu + L\omega,$$
$$G(x)\dot{x} + K(x)x = Hu + L(\omega),$$
$$\ddot{Q}M + QK(Q, \dot{Q}, u) = L(\omega),$$

where u and ω are control and disturbance variables, respectively, and $q \in \mathbb{R}^n$, $x \in \mathbb{R}^{2n}$, $Q \in \mathbb{R}^{3\times n/3}$.

Assume that the rigid bodies are rod-shaped and have negligible inertia about



their longitudinal axes.

Notation 1

- Let $\underline{e}_i, i = 1,2,3$ define a *dextral set* of unit vectors fixed in an inertial frame
- Define the *vectrix* $\underline{E}$ by $\underline{E} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{bmatrix}$.
- Two reference frames, $\underline{E}$ and $\underline{X}$. The transformation between these two frames is described by the 3×3 direction cosine matrix X^E so that $\underline{X} = \underline{E} X^E$.
- Let the 3×1 matrices r^X and r^E describe the components of the same vector $\underline{r}$ in the two reference frames $\underline{X}$ and $\underline{E}$,
- The relationship between the components of the same vector $\underline{r}$, described in two different reference frames, then

$$\underline{X} = \underline{E} X^E$$

$$\underline{r} = \underline{X} r^X = \underline{E} r^E = \underline{E} X^E r^X.$$

$$r^E = X^E r^X.$$

Notation 2

For any vector $v^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, define $\hat{v}$ and $\tilde{v}$ by

$$\hat{v} = \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix}, \qquad \tilde{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

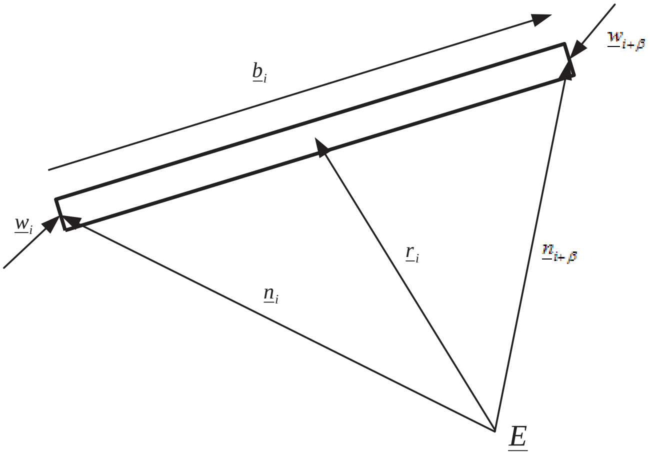
For any two vectors v, and x, $\hat{v}x = \hat{x}v$.
The cross product is $\underline{b}_i \times \underline{f}_j = (\underline{E}b_i) \times (\underline{E}f_j) = \underline{E}\tilde{b}_i f_j$,
The dot product is given by,
$\underline{b}_i \cdot \underline{f}_j = (\underline{E}b_i) \cdot (\underline{E}f_j) = b_i^T \underline{E}^T \cdot \underline{E}f_j = b_i^T f_j$,
where the dot product $\underline{E}^T \cdot \underline{E} = I$

Bar Vectors

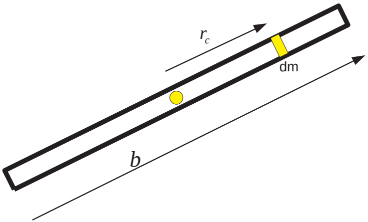
- System is composed of β bars and σ strings.
- Define the i^{th} node $\underline{n}_i$) of a structural system as a point on the rigid body at which strings are attached. The coordinates of this point of attachment is $n_i \in \mathbb{R}^3$.
- Define the i^{th} *string* as a massless structural member connecting two nodes. The vector connecting these two nodes is $\underline{s}_i$.
- Define the i^{th} *bar* of a β-bar system as a rigid rod connecting two nodes $\underline{n}_i$ and $\underline{n}_{i+\beta}$.
- The vector along the bar connecting nodes $\underline{n}_i$ and $\underline{n}_{i+\beta}$ is $\underline{b}_i = \underline{n}_{i+\beta} - \underline{n}_i, i = 1, 2, \ldots \beta$. The bar $\underline{b}_i$ has *length* $\|b_i\| = \mathscr{L}_i = \sqrt{b_i^T b_i}$.
- The vector $\underline{r}_i$ locates the mass center of bar $\underline{b}_i$, and $\underline{r}_i = \underline{E} r_i$.
- The vector $\underline{t}_i$ represents the force exerted on a node by string $\underline{s}_i$, where the direction of vector $\underline{t}_i$ is parallel to string vector $\underline{s}_i$. That is $t_i = \gamma_i s_i$ for some positive scalar γ_i.
- The *force density* γ_i in string s_i is defined by $\gamma_i = \frac{\|t_i\|}{\|s_i\|}$.

Bar force definition

The vector $\underline{f}_i$ represents the net sum of vector forces external to bar $\underline{b}_i$ terminating at node $\underline{n}_i$. The set of all nodal forces external to the bar $\underline{b}_i$ is described by the figure.



Angular Momentum



Then the angular momentum of the bar b_i about the center of mass of bar b_i, expressed in the $\underline{E}$ frame, is $\underline{h}_i = \underline{E}h_i$, where,

$$h_i = \frac{m_i}{12}\tilde{b}_i\dot{b}_i.$$

Dynamics of a rigid bar

Bar vector $\underline{b}$ connects nodes $\underline{n}_1$ and $\underline{n}_2$, at which are applied Forces $\underline{f}_1$ and $\underline{f}_2$.
The translation of the mass center of bar $\underline{b}$, located at position $\underline{r}$ obeys

$$m\ddot{r} = f_1 + f_2$$

The rotation of bar b_i about it mass center obeys

$$\frac{m}{12}\tilde{b}\ddot{b} = \frac{1}{2}\tilde{b}(f_2 - f_1).$$

The proof follows from the above expression of angular momentum and Newton's second law,

$$\dot{\underline{h}} = \frac{1}{2}\underline{b} \times (\underline{f}_2 - \underline{f}_1), \qquad \underline{h} = \frac{m}{12}\underline{b} \times \dot{\underline{b}}.$$

Constrained Dynamics: Fixed Length Bars

For fixed bar length $\mathscr{L}$, then $b^T b - \mathscr{L}^2 = 0$. The first two time derivatives of the length constraint yield $b^T \dot{b} = 0$ and $b^T \ddot{b} = -\dot{b}^T \dot{b}$. Then the total system is

$$\begin{aligned} \frac{m}{12}\tilde{b}\ddot{b} &= \frac{1}{2}\tilde{b}(f_2 - f_1) \\ b^T \ddot{b} &= -\dot{b}^T \dot{b} \\ b^T \dot{b} &= 0 \\ b^T b - \mathscr{L}^2 &= 0. \end{aligned}$$

Computational errors: Correct the length and velocity of each bar after each integration step, to keep the length constant and the velocity vector $\dot{b}$ perpendicular to the vector b.
At numerical iteration k, the computed values are $b(k)$ and $\dot{b}(k)$. Replace these values by the values $b(k+1)$ and $\dot{b}(k+1)$, given by

$$\begin{aligned} b(k+1) &= b(k)\mathscr{L} / \parallel b(k) \parallel \\ \dot{b}(k+1) &= \left(I - \frac{b(k+1)b^T(k+1)}{\mathscr{L}^2} \right) \dot{b}(k). \end{aligned}$$

Ignoring roundoff errors

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix} \ddot{b} = \begin{bmatrix} \tilde{b}(f_2 - f_1)\frac{6}{m} \\ -\dot{b}^T \dot{b} \end{bmatrix},$$

Solving for $\ddot{b}$ is a linear algebra problem. Uniqueness is guaranteed by the facts

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}^T \begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix} = \mathscr{L}^2 I,$$

,

$$\tilde{b}^2 = bb^T - b^T bI.$$

The unique Moore-Penrose inverse of $\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}$ is

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix}^+ = \begin{bmatrix} -\tilde{b} & b \end{bmatrix} \mathscr{L}^{-2}$$

,

Using the fact $\tilde{b}b = 0$, the unique solution for $\ddot{b}$ is

$$\begin{aligned}
\ddot{b} &= \frac{6}{m}(f_2 - f_1) - b\left(\frac{\dot{b}^T\dot{b}}{\mathscr{L}^2} + \frac{6}{m\mathscr{L}^2}b^T(f_2 - f_1)\right) \\
&= 6/m\left(I - \frac{bb^T}{\mathscr{L}^2}\right)(f_2 - f_1) - b\frac{\|\dot{b}\|^2}{\mathscr{L}^2}
\end{aligned}$$

The translational and rotational dynamics for a rod are given by

$$\begin{aligned}
\ddot{r} &= (f_1 + f_2)/m, \\
\ddot{b} + kb &= 6/m\left(I - \frac{bb^T}{\mathscr{L}^2}\right)(f_2 - f_1), \qquad k = \frac{\dot{b}^T\dot{b}}{\mathscr{L}^2}.
\end{aligned}$$

Dynamics of the Network of β bars, $b_i, i = 1, 2, \beta$

$$\ddot{b}_i \frac{m}{12} + b_i \theta_i = \frac{1}{2}(f_{\beta+i} - f_i)$$
$$m_i \ddot{r}_i = (f_i + f_{\beta+i}),$$

where

$$\theta_i = \frac{m_i}{12\mathscr{L}_i^2} \|\dot{b}_i\|^2 + \frac{b_i^T (f_{\beta+i} - f_i)}{2\mathscr{L}_i^2}$$

Now characterize the tendon forces, f_i and $f_{\beta+i}$

Matrix forms for the Network

Define matrices

$$
\begin{aligned}
F = \left[\begin{array}{cc} F_1 & F_2 \end{array}\right] &= \left[\begin{array}{cccc|ccc} f_1 & f_2 & \dots & f_\beta & f_{\beta+1} & \dots & f_{2\beta} \end{array}\right] \\
N = \left[\begin{array}{cc} N_1 & N_2 \end{array}\right] &= \left[\begin{array}{cccc|ccc} n_1 & n_2 & \dots & n_\beta & n_{\beta+1} & \dots & n_{2\beta} \end{array}\right] \\
T &= \left[\begin{array}{cccc} t_1 & t_2 & \dots & t_\sigma \end{array}\right] \\
S &= \left[\begin{array}{cccc} s_1 & s_2 & \dots & s_\sigma \end{array}\right] \\
B &= \left[\begin{array}{cccc} b_1 & b_2 & \dots & b_\beta \end{array}\right] \\
R &= \left[\begin{array}{cccc} r_1 & r_2 & \dots & r_\beta \end{array}\right] \\
\hat{\gamma} &= \mathrm{diag}\left[\begin{array}{ccc} \gamma_1 & \dots & \gamma_\sigma \end{array}\right],
\end{aligned}
$$

Then, $F \in \mathbb{R}^{3\times 2\beta}$, $N \in \mathbb{R}^{3\times 2\beta}$, $T \in \mathbb{R}^{3\times \sigma}$, $S \in \mathbb{R}^{3\times \sigma}$, $B \in \mathbb{R}^{3\times \beta}$, $\gamma \in \mathbb{R}^{\sigma}$, and

$$B = N_2 - N_1 = N \left[\begin{array}{c} -I \\ I \end{array}\right] = NU,$$

$$R = N_1 + \frac{1}{2}B.$$

$$T = S\hat{\gamma}.$$

Matrix Forms of Dynamics

For a β-bar system define the **configuration matrix** $\mathscr{Q}$ and,

$$\mathscr{Q} = \begin{bmatrix} B & R \end{bmatrix}$$

$$K_0 = \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{\theta} \begin{bmatrix} I & 0 \end{bmatrix}$$

$$\theta_i = b_i^T (f_{\beta+i} - f_i)/2\mathscr{L}_i^2 + m_i \|\dot{b}_i\|^2/12\mathscr{L}_i^2$$

$$\Phi = \begin{bmatrix} -\frac{1}{2}I & I \\ \frac{1}{2}I & I \end{bmatrix}$$

$$M = \mathrm{diag} \begin{bmatrix} \dots & m_i & \dots \end{bmatrix}$$

$$\mathscr{M} = \begin{bmatrix} \frac{1}{12}M & 0 \\ 0 & M \end{bmatrix}.$$

Then the dynamics for any class 1 tensegrity system are given by

$$\ddot{\mathscr{Q}}\mathscr{M} + \mathscr{Q}K_0 = F\Phi,$$

where $\mathscr{Q}\Phi^T = N$.

Bar Forces

For a given square matrix J, define the notation $\lfloor J \rfloor = \operatorname{diag}\begin{bmatrix} \dots & J_{ii} & \dots \end{bmatrix}$. Then, a matrix form for entries θ_i, $i = 1, 2, \dots$ is

$$\hat{\theta} = \frac{1}{2}\hat{\mathscr{L}}^{-2}\lfloor B^T(F_2 - F_1) + \frac{1}{6}\dot{B}^T\dot{B}M\rfloor$$

$$= \frac{1}{2}\hat{\mathscr{L}}^{-2}\lfloor\begin{bmatrix} I & 0 \end{bmatrix}\mathscr{Q}^T(F_2 - F_1) + \frac{1}{6}\begin{bmatrix} I & 0 \end{bmatrix}\dot{\mathscr{Q}}^T M\rfloor$$

$$\mathscr{L} = \begin{bmatrix} \mathscr{L}_1 & \mathscr{L}_2 & \cdots & \mathscr{L}_\beta \end{bmatrix}^T.$$

Connectivity Matrices

Define the **string connectivity matrix** C

$$C_{ij} = \begin{cases} 1 & \text{if string vector } s_i \text{ terminates on node } n_j. \\ -1 & \text{if string vector } s_i \text{ eminates from node } n_j. \\ 0 & \text{if string vector } s_i \text{ does not connect with node } n_j. \end{cases}$$

For 2β disturbance vectors applied at each of the β nodes,

$$W = \begin{bmatrix} w_1 & w_2 \cdots & w_{2\beta} \end{bmatrix}$$

Then

$$S = NC^T$$

,

$$TC = S\hat{\gamma}C = NC^T\hat{\gamma}C$$
$$F = W - TC = W - \mathscr{Q}\Phi^T C^T \hat{\gamma} C,$$

Final Form of Dynamics

For all Class 1 tensegrity systems with rigid bars,

$$\ddot{\mathscr{Q}}\mathscr{M} + \mathscr{Q}\mathscr{K} = W\Phi, \qquad \mathscr{Q} = \begin{bmatrix} B & R \end{bmatrix},$$

$$\mathscr{M} = \begin{bmatrix} \frac{1}{12}M & 0 \\ 0 & M \end{bmatrix}, \qquad M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_\beta \end{bmatrix}$$

$$\mathscr{K} = \begin{bmatrix} \hat{\theta} & 0 \\ 0 & 0 \end{bmatrix} + \Phi^T C^T \hat{\gamma} C \Phi, \qquad \Phi^T = \begin{bmatrix} -\frac{1}{2}I & \frac{1}{2}I \\ I & I \end{bmatrix}$$

$$\hat{\theta} = \frac{1}{12}\hat{\mathscr{L}}^{-2}\left\lfloor 6\begin{bmatrix} I & 0 \end{bmatrix}\mathscr{Q}^T(\mathscr{Q}\Phi^T C^T \hat{\gamma} C - W)\begin{bmatrix} I \\ -I \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix}\dot{\mathscr{Q}}^T\dot{\mathscr{Q}}\begin{bmatrix} I \\ 0 \end{bmatrix}M \right\rfloor.$$

Alternate Expressions

The i^{th} element of the diagonal matrix $\hat{\theta}$ may also be written

$$\theta_i = \frac{1}{2}\mathscr{L}_i^{-2} b_i^T(\mathscr{Q}\Phi^T C^T \hat{C}_{\Delta_i}\gamma - W_{\Delta_i}) + \frac{m_i}{12\mathscr{L}_i^2}||\dot{b}_i||^2,$$

$$\begin{aligned} W &= [W_1, W_2], \qquad C = [C_1, C_2] \\ W_{\Delta_i} &= i^{th} col(W_1 - W_2) = -W(i^{th} col(U)) = w_i - w_{\beta+1} \\ C_{\Delta_i} &= i^{th} col(C_1 - C_2) = -C(i^{th} col(U)) = C_i - C_{\beta+1}. \end{aligned}$$

Using $N = \mathscr{Q}\Phi^T$ it is straightforward to write

Theorem

Dynamics for all class 1 tensegrity in nodal coordinates N,

$$\ddot{N}M_N + NK_N = W,$$

where

$$M_N = \frac{1}{6}\begin{bmatrix} 2M & M \\ M & 2M \end{bmatrix}, \qquad K_N = U\hat{\theta}U^T + C^T\hat{\gamma}C,$$

where,

$$\theta_i = \frac{1}{2L_i^2}(NU)_i^T(NC^T\hat{C}_{\Delta_i}\gamma - WU_i) + \frac{m_i}{12L_i^2}||(\dot{N}U)_i||^2,$$

where $U_i = i^{th} col(U)$.

Statics

Replace $\mathscr{Q}(t)$ in the dynamic equations by its steady state value $Q = \lim t \to \infty[\mathscr{Q}(t)]$.
Then all equilibria for Class 1 tensegrity systems satisfies

$$QK = W\Phi, \qquad Q = \begin{bmatrix} B & R \end{bmatrix},$$

$$K = \begin{bmatrix} \hat{\theta} & 0 \\ 0 & 0 \end{bmatrix} + \Phi^T C^T \hat{\gamma} C \Phi, \qquad \Phi^T = \begin{bmatrix} -\frac{1}{2}I & \frac{1}{2}I \\ I & I \end{bmatrix}$$

$$\hat{\theta} = \frac{1}{12}\hat{\mathscr{L}}^{-2}\left\lfloor 6\begin{bmatrix} I & 0 \end{bmatrix}\mathscr{Q}^T(\mathscr{Q}\Phi^T C^T \hat{\gamma} C - W)\begin{bmatrix} I \\ -I \end{bmatrix}\right\rfloor.$$

These equations are LINEAR in the variable γ
Given a desired configuration Q and a set of static loads W **all** forces in the strings that are compatible with that configuration and load, are immediately known, by solving a linear algebra problem.

Vector Form of the Dynamics

$$\Gamma_b = \begin{bmatrix} 12 m_1^{-1} \mathscr{Q} \Psi^T \hat{\Psi}_1 \\ \vdots \\ 12 m_\beta^{-1} \mathscr{Q} \Psi^T \hat{\Psi}_\beta \end{bmatrix}, \quad \Gamma_r = \begin{bmatrix} m_1^{-1} \mathscr{Q} \Psi^T \hat{\Psi}_{\beta+1} \\ \vdots \\ m_\beta^{-1} \mathscr{Q} \Psi^T \hat{\Psi}_{2\beta} \end{bmatrix}$$

$$\Psi = \begin{bmatrix} I & 0 \\ -\frac{1}{2} C_\Delta & C_+ \end{bmatrix}, \quad \Lambda = \begin{bmatrix} (2L_1^2)^{-1} b_1^T \mathscr{Q} \Phi^T C^T \hat{C}_{\Delta_1} \\ \vdots \\ (2L_\beta^2)^{-1} b_\beta^T \mathscr{Q} \Phi^T C^T \hat{C}_{\Delta_\beta} \end{bmatrix}$$

$$\tau_b = 12 \begin{bmatrix} m_1^{-1}((w_{1+\beta} - w_1)/2 - \mathscr{Q} \Psi^T \hat{\Psi}_1 J \delta) \\ \vdots \\ m_\beta^{-1}((w_{2\beta} - w_\beta)/2 - \mathscr{Q} \Psi^T \hat{\Psi}_\beta J \delta) \end{bmatrix}$$

$$\tau_r = \begin{bmatrix} m_1^{-1}((w_{\beta+1} + w_1) - \mathscr{Q} \Psi^T \hat{\Psi}_{\beta+1} J \delta) \\ \vdots \\ m_\beta^{-1}((w_\beta + w_{2\beta}) - \mathscr{Q} \Psi^T \hat{\Psi}_{2\beta} J \delta) \end{bmatrix}$$

Vector Form of Dynamics

$$\delta = \begin{bmatrix} (12\mathscr{L}_1^2)^{-1} m_1 \|\dot{b}_1\|^2 + (2\mathscr{L}_1^2)^{-1} b_1^T (w_{1+\beta} - w_1) \\ \vdots \\ (12\mathscr{L}_\beta^2)^{-1} m_\beta \|\dot{b}_\beta\|^2 + (2\mathscr{L}_\beta^2)^{-1} b_\beta^T (w_{2\beta} - w_\beta) \end{bmatrix}$$

$$q^T = \begin{bmatrix} b_1^T & \cdots & b_\beta^T & r_1^T & \cdots & r_\beta^T \end{bmatrix}, \quad \mathscr{Q} = \begin{bmatrix} E_1 q & \cdots & E_{2\beta} q \end{bmatrix}$$

$$b_i = \mathscr{Q} e_i, \; E_i = \begin{bmatrix} 0 & \cdots & I_3 & \cdots & 0 \end{bmatrix}, \; e_i^T = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}.$$

Theorem

All class 1 tensegrity dynamics satisfy

$$\ddot{q} + \Gamma G \gamma = \tau,$$

$$\Gamma = \begin{bmatrix} \Gamma_b \\ \Gamma_r \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_b \\ \tau_r \end{bmatrix}, \quad G = \begin{bmatrix} \Lambda \\ I \end{bmatrix}.$$

Proof.

the vector θ is given by

$$\theta = \Lambda\gamma + \delta,$$

and the matrix $\mathscr{K}$, and its i^{th} column are given by

$$\mathscr{K} = \Psi^T \hat{u} \Psi$$

$$\mathscr{K}_i = \Psi^T \hat{\Psi}_i (G\gamma + J\delta)$$

$$u = \begin{bmatrix} \theta \\ \gamma \end{bmatrix} = \begin{bmatrix} \Lambda \\ I_\sigma \end{bmatrix} \gamma + \begin{bmatrix} I_\beta \\ 0 \end{bmatrix} \delta = G\gamma + J\delta$$

□

Now let $t \to \infty$ in theorem above.
Define the left null-space of a matrix $[\cdot]$ by
$({}^{\perp}[\cdot])[\cdot] = 0$,
Define the right-nullspace of matrix $[\cdot]$ by
$[\cdot]([\cdot]^{\perp}) = 0$
Define the Moore-Penrose inverse of the matrix $[\cdot]$ by $[\cdot]^{\dagger}$
A specified set of constant loads W is said to be **admissible** for a specified $\mathscr{Q}$ if there exists a solution γ to the equation $\Gamma G \gamma = \tau$. That is, if

$${}^{\perp}[\Gamma G(\mathscr{Q})]\tau(\mathscr{Q}, W) = 0.$$

For statics $\Gamma G \gamma = \tau$, then the string force densities of all static equilibria of all class 1 tensegrity structures are given by

$$\gamma = [\Gamma G]^{\dagger}\tau + [\Gamma G]^{\perp} z,$$

where z is arbitrary, subject to the requirement that all elements of γ are positive or zero.

Conclusions

- The nonlinear dynamics are given in the form of a second order differential equation of a $3 \times 2\beta$ *configuration matrix*, $\mathscr{Q}$.
- The matrix form of dynamics has the simplest form, even though they are non-minimal (5β degree-of-freedom system is modeled by 6β degrees of freedom)
- The equations contain no trigonometric nonlinearities, and the mass matrix is constant.
- The simplicity of these equations is partly due to the use of the matrix form, partly due to the choice of variables, and partly due to the enlarged space in which the dynamics are described.
- Numerical integration schemes for stabilizing the computational errors are given.
- All static equilibria are characterized. For any given admissible configuration the equilibria equations are linear in the string force densities.
- The freedom in the desired equilibria can be utilized to further reduce control effort in the control problem.

Motivation and Goals for Control

- Tensegrity systems can have many rigid bodies connected either by elastic "springs", or "strings".
- This lecture assumes springs are used
- Do this with a non-minimal realization of the dynamic model
- Applications: formation flying such as a fleet of rigid vehicles.
- The elastic "springs" connecting these rigid bodies could then be the result of closed loop control laws that mimic the spring and damper forces of the "interconnections" of the vehicles
- We provide state feedback control laws to modify a configuration (formation) from some initial known configuration to a desired final configuration, using the smallest control effort.
- Potential advantage: exploit the special structure of the equations to obtain an efficient simulation, or control law

Control Strategy For Bilinear Systems

,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}, \qquad \mathbf{y} = \mathbf{g}(\mathbf{x})$$

- Define a **vector** of Lyapunov-type functions, $\mathbf{y}$
- Find control laws that force this vector to behave as a linear stable ODE.
- The controllers required to do this are nonlinear.
- We do **not** seek to make the dynamics of the closed loop plant linear.
- We force nonlinear Lyapunov-type functions to behave linearly.
- The Lyapunov-type functions are not required to be quadratic, although such examples are given here

- Let $\mathbf{y}$ represent a chosen vector of nonlinear functions that are important to reduce toward zero (such as the error squared between current and desired configurations, etc).
- For example, each element of the vector $\mathbf{y}$ could be chosen such as $y_i = \mathbf{x}^T \mathbf{Q_i} \mathbf{x}$, where $\mathbf{Q_i}$ is chosen to create an error function that should be zeroed by the control system.
- If $\mathbf{y}$ represents the difference between the current and a desired "output", then a desired state is achieved by forcing y_i to toward zero. One way to reduce $\mathbf{y}$ toward zero is to force $\mathbf{y}$ to satisfy a stable linear differential equation,
- $\dot{\mathbf{y}} = -\mathbf{\Omega} \mathbf{y}$, where $\mathbf{\Omega}$ is chosen positive definite.
- The method does not require quadratic choices for y_i. (Indeed, some nonlinear systems cannot be stabilized with quadratic Lyapunov functions).
- If it is not possible to satisfy $\dot{\mathbf{y}} = -\mathbf{\Omega} \mathbf{y}$ exactly, then we will minimize the Euclidean norm of the error $\dot{\mathbf{y}} + \mathbf{\Omega} \mathbf{y}$ at each instant of time. This yields the first result:

Derivation of the ORC

Find a control $\mathbf{u}$ which minimizes the Euclidean norm squared of the error $\dot{\mathbf{y}} + \boldsymbol{\Omega}\mathbf{y}$, while
choosing $\boldsymbol{\Omega}$ to minimize an integral of the error over a finite interval of time (t_1, t_2).
Note that

$$\dot{\mathbf{y}} = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}) = -\boldsymbol{\Omega}\mathbf{g}(\mathbf{x}).$$

Linear algebra yields the minimal Euclidean norm squared of the error in the above equation (left-hand side minus the right-hand side) yields

$$\mathbf{u} = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)^{+}(\boldsymbol{\Omega}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}),$$

Compute the Euclidean norm squared of the equation error, J_1,

$$J_1 = (\boldsymbol{\Omega}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f})^T\mathbf{P}(\boldsymbol{\Omega}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f})$$

$$\mathbf{P} := \mathbf{I} - \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)^{+}.$$

The ORC controller

Theorem

For the system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x})$$

the control law given by (the ORC controller)

$$\mathbf{u} = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)^{+}(\boldsymbol{\Omega}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}),$$

minimizes, at each instant of time, the Euclidean norm squared of the error $\dot{\mathbf{y}} + \boldsymbol{\Omega}\mathbf{y}$.

For insight lets examine the use of this ORC controller in a *linear* system.

The ORC Controller Applied to a Linear System

Theorem

For any linear system such that $\mathbf{y} = \mathbf{g}(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}\mathbf{x}$, *and* $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, *if* $\mathbf{Q}$ *is chosen to satisfy*

$$\mathbf{Q}\mathbf{A} + \mathbf{A}^T\mathbf{Q} - 2\mathbf{Q}\mathbf{B}\mathbf{B}^T\mathbf{Q} + \Omega\mathbf{Q} = \mathbf{0},$$

then the ORC controller minimizes J *given by*

$$J = \int_0^\infty (\Omega\mathbf{x}^T\mathbf{Q}\mathbf{x} + \frac{1}{2}\mathbf{u}^T\mathbf{u})dt,$$

and the closed loop system has these properties

$$\dot{\mathbf{y}} = -\boldsymbol{\Omega}\mathbf{y}$$
$$Min(J) = \mathbf{x_0}^T\mathbf{Q}\mathbf{x_0}$$

Proof.
Computing the control in the linear case yields

$$\mathbf{u} = -\frac{\mathbf{x}^T(\Omega\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{A}^T\mathbf{Q})\mathbf{x}}{2\mathbf{x}^T\mathbf{Q}\mathbf{B}\mathbf{B}^\mathbf{T}\mathbf{Q}\mathbf{x}}\mathbf{B}^T\mathbf{Q}\mathbf{x},$$

where the fraction reduces to 1 when $\mathbf{Q}$ satisfies the Riccati above, which is indeed the condition for minimizing J. $\square$

- The linear case the ORC controller is optimal by **two** different criteria.
- It is the least squares solution which minimizes, at any instant of time, the Euclidean norm of the error $\dot{\mathbf{y}} + \mathbf{\Omega}\mathbf{y}$
- It is also an optimal LQR control for the functional J
- These results provide encouragement to investigate further properties of the ORC controller in the nonlinear case.
- For the linear case there is no restriction on the choice of positive matrix $\mathbf{\Omega} > \mathbf{0}$, since the equation $\dot{\mathbf{y}} = -\mathbf{\Omega}\mathbf{y}$ is always solved
- In the nonlinear case the controller might not exist that satisfies $\dot{\mathbf{y}} = -\mathbf{\Omega}\mathbf{y}$. In this case we make a special choice of $\mathbf{\Omega}$ in an attempt to reduce error.

Now lets evaluate the error $J_2 = \int J_1 dt$ over an interval of time (t_1, t_2),
J_2 is the inner product of the vector $\mathbf{P}(\mathbf{\Omega g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f})$ with itself.
Note the property of the projection matrix $\mathbf{P}^2 = \mathbf{P}$
Recall the notation and definitions of inner and outer products, on a given interval, (t_1, t_2),

$$< \mathbf{x}, \mathbf{y} > = \int_{t_1}^{t_2} \mathbf{x}^T(t)\mathbf{y}(t)dt$$

$$> \mathbf{x}, \mathbf{y} < = \int_{t_1}^{t_2} \mathbf{x}(t)\mathbf{y}^T(t)dt$$

$$< \mathbf{x}, \mathbf{y} > = tr(> \mathbf{x}, \mathbf{y} <).$$

From functional analysis, we have the standard known result:

Lemma

The constant $\mathbf{A}$ *which minimizes the inner product* $< \mathbf{Ax} - \mathbf{b}, \mathbf{P}(\mathbf{Ax} - \mathbf{b}) >$ *is* $\mathbf{A} = (> \mathbf{b}, \mathbf{x} <)(> \mathbf{x}, \mathbf{x} <)^{-1}$.

Applying this result to our problem of choosing $\mathbf{\Omega}$ to minimize the inner product $< \mathbf{P}(\mathbf{\Omega g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}), \mathbf{P}(\mathbf{\Omega g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}) >$ yields

$$\mathbf{\Omega}(t_1,t_2) = -\mathbf{P}\left(\int_{t_1}^{t_2} \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}\mathbf{g}^T\right)\left(\int_{t_1}^{t_2} \mathbf{g}\mathbf{g}^T\,dt\right)^{-1} + (\mathbf{I}-\mathbf{P})\mathbf{Z},$$

where $\mathbf{P} = \mathbf{P}^2 = \mathbf{P}^+$, and $\mathbf{Z}$ is an arbitrary matrix. At time t_2 the control switches to,

$$\mathbf{u}(t_1,t_2) = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)^+ \left(\mathbf{\Omega}(t_1,t_2)\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}\right),$$

This control function switches to $\mathbf{u}(t_2,t_3)$, at time t_3, etc. So the nonlinear controller periodically updates the control with a new $\mathbf{\Omega}$, based upon a calculation from performance achieved over the past interval
Similar in spirit to Model Predictive Control, except
update the **performance requirement**, $\mathbf{\Omega}$, instead of a **model** , $\mathbf{A}$, of the plant,
The adaptive ORC procedure updates (updates the required stability properties of the chosen performance function $\mathbf{y}$

Summary of Adaptive ORC

Let t_k and t_{k+1} represent the times at which two consecutive updates of $\mathbf{\Omega}$ are made (the interval $t_{k+1} - t_k$ is not necessarily uniform). Then the control law can be written in the form

$$\mathbf{u}(t_{k+1}) = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)^{+}\left(\mathbf{\Omega}(t_k, t_{k+1})\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}\right),$$

where the control law $\mathbf{u}(t_{k+1})$ is applied on the interval $t_{k+1} - t_k$, and then the controller is updated to $\mathbf{u}(t_{k+2})$, and so forth. The initial controller for the first interval is

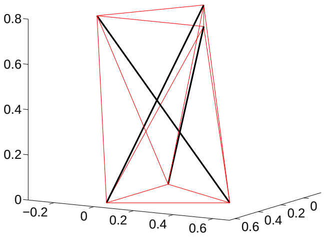
$$\mathbf{u}(t_0) = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)^{+}\left(\mathbf{\Omega}_0\mathbf{g} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}\right),$$

where $\mathbf{\Omega}_0$ is a selected positive definite matrix.

Exponential Stability of the ORC

Uniqueness of the ORC implies $\dot{\mathbf{y}} = -\boldsymbol{\Omega}\mathbf{y}$, which yields exponential stability of the output, since $\mathbf{y}(t) \leq e^{-\boldsymbol{\Omega}t}\mathbf{y}_0$.
The standard approach to exponential stability is the use a single Lyapunov function (y is a scalar in the case).However, the linear algebra problem allows a control nullspace of dimension $m-1$, leaving $m-1$ free variables in the selection of the m-dimensional control vector.
This leaves quite a bit of room to satisfy still more constraints, without compromising the original one. $V_i(\mathbf{x}) \rightarrow 0, i = 1, \cdots, k$. This can reduce the nullspace in the control problem to be no larger than dimension $|m-k|$.

Configuration Control of a Tensegrity Prism



Initial and final configurations for the movement performed with the 1 stage Prism

Dynamics, and State Errors

Dynamic model is

$$\ddot{\mathbf{q}} = \mathbf{B}_2(\mathbf{q})\gamma + \tau(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{w})$$

vector $\mathbf{q}$ contains the bar vectors $\mathbf{b_i}$ and their centers of mass $\mathbf{r_i}$
γ is the vector of control inputs,
The state vector will be given by the actual state minus the desired state:
$\mathbf{x}^T = ((\mathbf{x}_i)^T, ..., (\mathbf{x}_j)^T)$, for $i = 1,2,3.$ and $j = 1,2,3.$

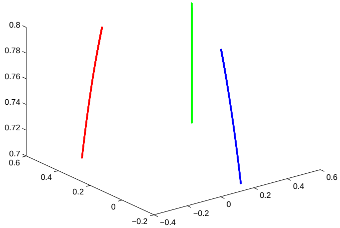
$$\mathbf{x}_i^T = ((\mathbf{b}_i - \mathbf{b}_{i_d})^T, (\dot{\mathbf{b}}_i)^T)$$
$$\mathbf{x}_j^T = ((\mathbf{r}_j - \mathbf{r}_{j_d})^T, (\dot{\mathbf{r}}_j)^T)$$

Example A:
$y = g(\mathbf{x}) = \mathbf{x}^T \mathbf{Q}\mathbf{x}, \qquad \Omega = \alpha > 0$
$y = \sum_i^3 \mathbf{x}_i^T \mathbf{Q}_i \mathbf{x}_i + \sum_j^3 \mathbf{x}_j^T \mathbf{Q}_j \mathbf{x}_j,$
$\mathbf{Q} = BlockDiag[...\mathbf{Q}_i.....\mathbf{Q}_j...]$
for $i, j = 1, 2, 3$ and $\Omega = 0.3$,
$\mathbf{T}_{b_i} = 2.5\mathbf{I}$, $\mathbf{X}_{b_i} = 0.7\mathbf{I}$, $\mathbf{Z}_{b_i} = 3\mathbf{I}$, $\mathbf{T}_{r_j} = 1.5\mathbf{I}$, $\mathbf{X}_{r_j} = \mathbf{I}$, $\mathbf{Z}_{r_j} = \frac{3}{2}\mathbf{I}$,
where

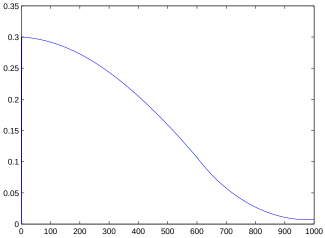
$$\mathbf{Q}_i = \begin{pmatrix} \mathbf{T}_{b_i} & \mathbf{X}_{b_i} \\ \mathbf{X}_{b_i}^T & \mathbf{Z}_{b_i} \end{pmatrix}$$

$$\mathbf{Q}_j = \begin{pmatrix} \mathbf{T}_{r_j} & \mathbf{X}_{r_j} \\ \mathbf{X}_{r_j}^T & \mathbf{Z}_{r_j} \end{pmatrix}$$

Reconfiguring the Tensegrity Prism



(a) Upper nodes trajectories



(b) Values for V

Figure: Ex.A. Left, upper nodes trajectories (node 4 green, 5 red and 6 blue). On the right, the value for the Lyapunov function.

Control Signals (string tensions)

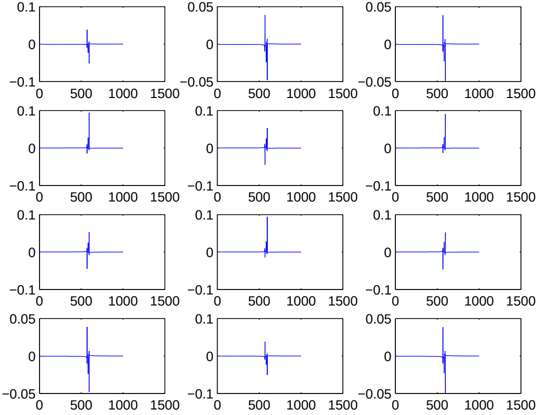


Figure: Ex.A. Control signals evolution. Tensions in cables are listed from top left to down right corresponding to cables 1 to 12.

Vector ORC

Example B: ($y_i = g_i(\mathbf{x}) = \mathbf{x}^T \mathbf{Q_i} \mathbf{x}, \Omega \geq 0$)
vector of functions $\mathbf{y}$ ($\mathbf{y}$ is 6x1)
Find the control to satisfy $\dot{\mathbf{y}} = -\boldsymbol{\Omega}\mathbf{y}$, for the choice $\boldsymbol{\Omega}_{ij} = \alpha\delta_{ij}$, hence, $\dot{V}_i = -\alpha V_i$.

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{pmatrix} \psi_{b_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \psi_{b_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \psi_{b_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \psi_{r_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \psi_{r_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \psi_{r_3} \end{pmatrix}$$

where each of the blocks is 1x3 and $\frac{\partial \mathbf{g}}{\partial \mathbf{x}}$ is 6x18,

$$\psi_{b_i} = (\mathbf{b}_i - \mathbf{b}_{i_d})^T \mathbf{Y}_{b_i} + \dot{\mathbf{b}}_i^T \mathbf{Z}_{b_i}$$
$$\psi_{r_i} = (\mathbf{r}_i - \mathbf{r}_{i_d})^T \mathbf{Y}_{r_i} + \dot{\mathbf{r}}_i^T \mathbf{Z}_{r_i}$$

$\boldsymbol{\Omega}$ is constant, diagonal, 6x6, and the decay rate is $\alpha = 0.01$.
components $\mathbf{Q}_i$:
$\mathbf{T}_{b_i} = 2.5\mathbf{I}, \mathbf{X}_{b_i} = 0.7\mathbf{I}, \mathbf{Z}_{b_i} = 3\mathbf{I}$, $\mathbf{T}_{r_i} = 1.5\mathbf{I}, \mathbf{X}_{r_i} = \mathbf{I}, \mathbf{Z}_{r_i} = \mathbf{I}$

Vector ORC Control Signals

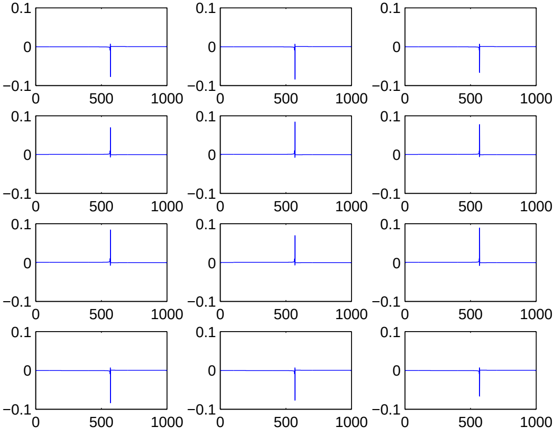
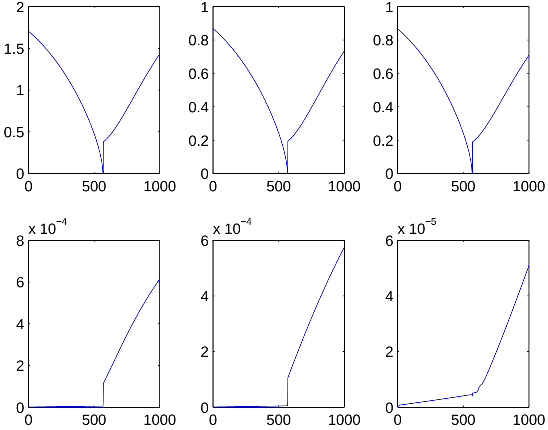


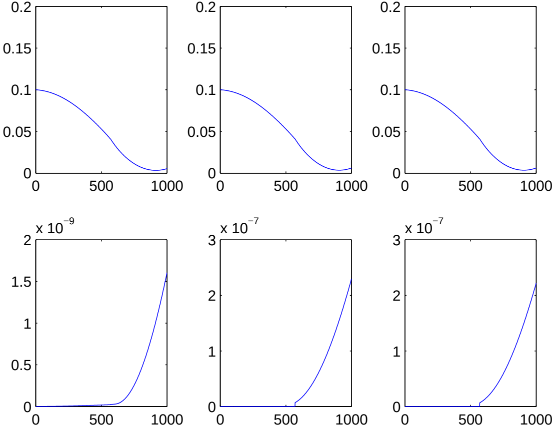
Figure: Ex.B. Control signals evolution. Tensions in cables are listed from top left to down right corresponding to cables 1 to 12.

Singular Values of the $\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)$ matrix



Singular values of $\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)$ matrix

Decay of Lyapunov Functions



Lyapunov functions values evolution (from left to right and top-down, $V_{b_1}, V_{b_2}, V_{b_3}, V_{r_1}, V_{r_2}, V_{r_3}$)

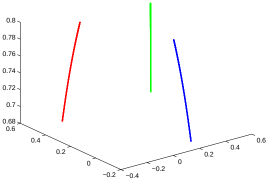
Example C:

$$y_i = g_i(\mathbf{x}) = \mathbf{x}^T \mathbf{Q_i} \mathbf{x},$$

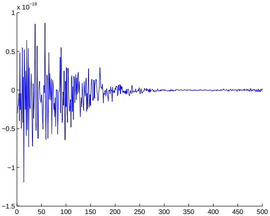
$$\Omega = -\mathbf{P}\left(\int_{t_1}^{t_2} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mathbf{f} \mathbf{g}^T\right)\left(\int_{t_1}^{t_2} \mathbf{g}\mathbf{g}^T dt\right)^{-1} + (\mathbf{I} - \mathbf{P})\mathbf{Z},$$

As in the previous example we have 6 different g_i. The matrix $\frac{\partial \mathbf{g}}{\partial \mathbf{x}}$ is 6x18. After an initial guess for $\mathbf{\Omega}$ it is recomputed between each interval of time.

Adaptive ORC Example, Trajectories

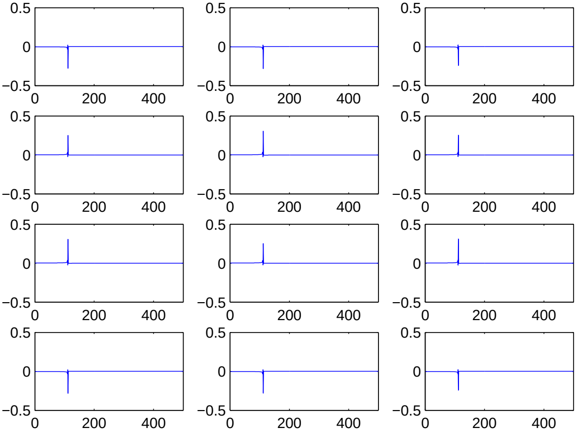


Upper nodes trajectories



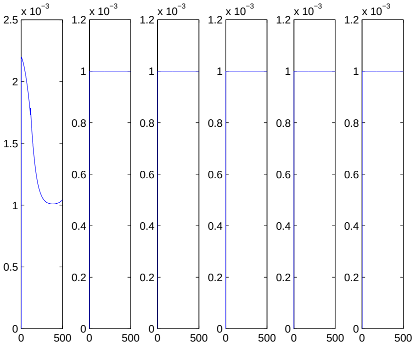
$J1$ error

Adaptive ORC: Control Trajectories



Control signals. Tensions in cables are listed from top left to down right corresponding to cables 1 to 12

Adaptive ORC: Singular values of matrix $\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{B}\right)$



Singular Values of the outer product of $\mathbf{g}$ during the simulation time

Conclusions

The ORC offers a multiple set of performance requirements, instead of just one V_i.
The eigenvalues of $\mathbf{\Omega}$ do not have to be real.
The ES system forces the V_i functions to monotonically reduce in time, and this is much more severe than just asking just one V_i to go to zero,
This severe requirement on the performance can cause the control effort to be unreasonably large.
The ability to choose the entire matrix $\mathbf{\Omega}$ instead of just its diagonal elements offers greater flexibility to guarantee a solution
The adaptive ORC offers a more clever choice of $\mathbf{\Omega}$.
It is ad hoc to find a good $\mathbf{\Omega}$.
Then ORC method chooses $\mathbf{\Omega}$ to minimize this error over a finite time interval (t_1, t_2).
In this sense, the method is similar in spirit to Model Predictive Control, except that no linear model need be computed (the update of $\mathbf{\Omega}$ plays that role).

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